67-18

GEOLOGICAL SURVEY OF CANADA

DEPARTMENT OF ENERGY, MINES AND RESOURCES

PAPER 67-18

A RE-EXAMINATION OF CERTAIN STATISTICAL METHODS IN PALAEOMAGNETISM

A. Larochelle

Price, 75 cents

1967

CANADA

GEOLOGICAL SURVEY OF CANADA

PAPER 67-18

A RE-EXAMINATION OF CERTAIN STATISTICAL METHODS IN PALAEOMAGNETISM

A. Larochelle

DEPARTMENT OF ENERGY, MINES AND RESOURCES

Available by mail from the Queen's Printer, Ottawa,
from the Geological Survey of Canada,
601 Booth St., Ottawa
and at the following Canadian Government bookshops:

OTTAWA
Daly Building, Corner Mackenzie and Rideau

TORONTO
221 Yonge Street

MONTREAL
Æterna-Vie Building, 1182 St. Catherine St. West

WINNIPEG
Mall Center Bldg., 499 Portage Avenue

VANCOUVER
657 Granville Street

or through your bookseller

A deposit copy of this publication is also available
for reference in public libraries across Canada

Price, 75 cents   Cat. No. M44-67-18

Price subject to change without notice

ROGER DUHAMEL, F.R.S.C.
Queen's Printer and Controller of Stationery
Ottawa, Canada
1967

CONTENTS

ABSTRACT

The particularities of statistical methods in the field of Palaeomagnetism are discussed. A novel approach is used to derive expressions giving estimates of the angular variance and of the angular standard deviation of a population on the basis of a sample of size N drawn randomly from the population. Tests of significance based on these definitions are described. These tests turn out to be identical to those previously described by Watson for the particular case where the vectors have a fisherian distribution. A numerical example is worked out to illustrate the application of the tests.

A RE-EXAMINATION OF CERTAIN STATISTICAL METHODS IN PALAEOMAGNETISM

INTRODUCTION

Statistics have long proven useful in many branches of modern experimental science and many developmentsin modern technology owe their origin to this branch of mathematics. In the particular domain of Palaeomagnetism, statistics began to be used on a wide basis a little more than a decade ago, after the publication of the classic papers by Fisher (1953), Vincenz and Bruckshaw (1954), Watson (1956 a, b) and Watson and Williams (1956). Unfortunately these papers are written in a language not readily understood by the non-statistician and too often their contentshave been misunderstood. An attempt to solve this problem was made in a paper by Watson and Irving (1957) but in the writer's opinion, much of the material in that paper is misleading and needs to be reassessed. It seems appropriate to do this now that the output of palaeomagnetic data from various laboratories throughout the world is constantly increasing. The object of the present paper will be to derive in words understood by the geologist and the geophysicist, the principles and methods which form the basis of statistical analysis in Palaeomagnetism.

PRESENTATION OF DATA

It would be beyond the scope of this paper to review in detail the techniques and instrumentation currently used in Palaeomagnetic research. It will suffice to say that palaeomagnetic inferences are based essentially on measurements of the remanent magnetization of rock specimens oriented in situ at the time of their sampling.

There are several factors which could contribute in the dispersion of either the intensities or directions of magnetization in a geological formation. Among them are the effects of lightning, the relative displacements of the rock in situ, the rock inhomogeneity, the secular variation of the earth's field at the time the rock was formed, etc. In a set of Palaeomagnetic directions obtained from N independently oriented samples of a formation, sampling and measurement errors will generally contribute to an increase in the angular dispersion.

To some extent it is possible to reduce the effects of some of the factors of dispersion or to allow for them but there will always remain an element of uncertainty as to the precision of the data. On the other hand, the various factors will generally partly militate against each other and their

Manuscript received: January 24, 1967.

resultant effect will normally spread out according to the laws of hazard: this is the justification for using statistics in Palaeomagnetism.

It should also be pointed out that remanent magnetizations in rocks are essentially vectorial quantities and, because of this, their study requires a double statistical analysis. The first deals with the dispersion of the moduli of vectors about their arithmetic mean whereas the second considers the angular dispersion of the same vectors about their mean direction. The first analysis does not call for any special approach other than those already available in standard textbooks on statistics. On the other hand, the statistics of directions being somewhat limited to a small number of fields, they are naturally less widely discussed and, for this reason, this paper will deal mostly with this aspect of statistics in Palaeomagnetism.

The complex nature of palaeomagnetic data implies naturally that measurements be carried out on a sufficient number of samples. The sampling of a formation will be carried out, for example, at different stratigraphic levels and at many different localities where careful note will be made of the present attitude of the formation. At each collecting site, two or more samples will be oriented separately in order to eliminate as much as possible the errors inherent in the collecting technique or those brought in by a number of possible local factors. The magnetization measurements will generally be done on two or more specimens cut from each sample in order to reduce the chances of gross error during measurements. The mean direction of magnetization of a sample will be taken as the mean direction of the palaeomagnetic vectors obtained for all its specimens, provided the latter are not diverging significantly.

If the sampling is extended over several sites a large number of sample mean directions will result and these may be listed in the form of a table or represented graphically as dots on a stereographic or equal-area projection net. The few qualitative conclusions that may be drawn from such a body of data will generally be limited however and it will be necessary to synthesize the data in a form which will permit, for instance, a comparison between sets of data or the separation of the various factors of dispersion in a specific set.

Independently of the hierarchical level of the sampling, a group of palaeomagnetic vectors may be represented by a small number of characteristics or statistical parameters. Among the most useful in statistical analysis are the mean direction, the angular standard deviation and the angular variance. I shall first define these terms and derive the equations which will serve in estimating their value for a population on the basis of the information given by a representative sample of the population.

Assume a group of N vectors sampled at random from a population of which it is proposed to estimate the mean direction, the angular variance, and the angular standard deviation. Their mean direction is obtained after having reduced to unity the modulus of all N vectors by calculating their resultant R. The direction cosines (l, m, n) of this resultant define the mean direction of the sample.

The angular deviation between two vectors is defined simply by the angle between them. The mean angular deviation of the N vectors with respect to the mean direction of the population may thus be defined as

$$\bar{\delta} = \frac{\sum \delta_i}{N} \tag{1}$$

where δ_i is the angular deviation of the i^{th} vector from the population mean direction.

The angular variance and the angular standard deviation, by analogy with their counterpart in standard statistics, σ^2 and σ , are defined by

$$\delta^2 = \frac{\sum \delta_i^2}{N} \tag{2}$$

$$\delta = \left(\frac{\sum \delta_i^2}{N} \right)^{1/2} \tag{3}$$

In practice, the values of the δ_i are unknown as the mean direction of the population is itself unknown. However, Fisher (1953) has shown that the best estimate of this direction obtainable from a sample is defined by the direction cosines of the resultant (l, m, n). If we call λ, μ and ν the direction cosines of the population mean, we may express the angle $\delta_{\bar{R}}$ between the mean direction of the sample and that of the population by

$$\cos \delta_{\bar{R}} = \frac{\lambda \sum l_i + \mu \sum m_i + \nu \sum n_i}{R} \tag{4}$$

where l_i, m_i and n_i are the direction cosines of the i^{th} vector. The minimum value of $\delta_{\bar{R}}$ implies that the two vectors be parallel, i.e. that

(5) $$\frac{R\lambda}{\sum l_i} = \frac{R\mu}{\sum m_i} = \frac{R\nu}{\sum n_i}$$

The estimation of the angular variance of the population is somewhat more complex as the angular dispersion of the population consists of (a) the within-sample dispersion and (b) the dispersion of its mean with respect to the population, i.e. $\delta_{\bar{R}}$. Assuming that

$$\frac{\delta_i^4}{4!} \ll 1 - \frac{\delta_i^2}{2}$$

for any vector in the sample, then

(6) $$\cos \delta_i = 1 - \frac{\delta_i^2}{2}$$

and

(7) $$\begin{aligned} \sum \cos \delta_i &= \lambda \sum l_i + \mu \sum m_i + \nu \sum n_i \\ &= R \cos \delta_{\bar{R}} = R\left(1 - \frac{\delta_{\bar{R}}^2}{2}\right) \\ &= N - \sum\left(\frac{\delta_i^2}{2}\right) \end{aligned}$$

which, combined with (2) yields

$$\frac{N\delta^2}{2} = N - R\left(1 - \frac{\delta_{\bar{R}}^2}{2}\right) \tag{8}$$

A well-known equation (see e.g. Kemptone, 1951) in the theory of the Laplace-Gauss normal distribution establishes the relationship between the variance $\sigma_{\bar{x}}^2$ of the mean $\bar{x}$ of a sample and the variance σ^2 of the population represented by the sample. We may assume that the analogue of this equation,

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{N} \tag{9}$$

is, in vector statistics,

$$\delta_{\bar{R}}^2 = \frac{\delta^2}{R} \tag{10}$$

which, if introduced into (8) yields

$$\hat{\delta}^2 = \frac{2(N-R)}{(N-1)} \tag{11}$$

where the symbol $\hat{\delta}^2$ is used to indicate that the right-hand side of (11) is an estimate of the population variance. In the same way, an estimate of the population angular standard deviation is given by

$$\hat{\delta} = \left(\frac{2(N-R)}{(N-1)} \right)^{1/2} \tag{12}$$

and an estimate of the angular standard deviation of the mean is given by

$$\hat{\delta}_{\bar{R}} = \hat{\delta} R^{-1/2} \tag{13}$$

The justification of (10) comes from the fact that if a sample of N unit vectors having a resultant R is tightly grouped about its mean, the probability that the latter does not diverge appreciably from the population mean would likely be greater than if the sample is highly scattered. Because the standard deviation of the sample tends toward zero when the length of the resultant tends toward N and this, independently of N, it appears that the role played by N in (9) is played by R in (10). The same reasoning may be used to suggest that the right-hand side of (13) represents a more accurate estimate of the angular standard deviation of the mean than the expressions $(2/kN)^{1/2}$ and $81°/(kN)^{1/2}$ which are currently used in the Palaeomagnetic literature. It is noted furthermore that a mathematical derivation of these last two expressions has never been given.

STATISTICAL INTERPRETATION

The possibility of defining the distribution of a population of unit vectors by an estimate of its mean direction and of its angular variance will only be useful if the population is not randomly oriented or isotropic. In any particular study, it is thus important to verify this condition as the results of the statistical analysis of an isotropic population would in general be misleading. The criterion used to detect the isotropy of a population is based on the modulus of the resultant R of the sample supposedly drawn at random from the population. If this modulus is greater than a limit value which we will call R_o, it will be concluded that the population has a probability P of being isotropic.

It would be beyond the scope of this paper to review the methods suggested by Rayleigh (1919), Vincenz and Bruckshaw (1960) and others for the calculation of R_o. Watson (1956 b) has derived, on a rigid mathematical basis, a method of calculating R_o and Stephens (1964) has used this method to calculate R_o for values of N from 4 to 25 and this, at probability levels of .01, .02 and .05. He has also given an equation to calculate approximate values of R_o for any number N and at any probability level.

Once it has been established that the population dealt with is not isotropic, a comparison of various groups of vectors may be made from the point of view of their dispersions. To do this the concept of degrees of freedom which is fundamental in dealing with this problem will be introduced.

Assuming a series of values such that

$$\sum_{i=1}^{k} x_i = S \tag{14}$$

where S has a fixed value, the left-hand member of (14) is said to have (k-1) degrees of freedom because, if any value is assigned to each of the first (k-1) x_i's, that of the k^{th} is specified by the equation. Similarly, if R is the length of the resultant of a set of N unit vectors, the quantity (N-R) has 2(N-1) degrees of freedom because any one set of (N-1) vectors among the N vectors (which normally have 2 degrees of freedom each) may assume an infinite number of configurations and the end of their resultant always falls at a given point on the surface of a unit sphere centred at the extremity of R. On the other hand, the attitude of the N^{th} vector is fixed by the conditions of the problem and therefore has no degree of freedom. As the variance estimate of a population represented by a set of N vectors is proportional to (N-R), it is therefore based on 2(N-1) degrees of freedom.

A test known as Snedecor's F-ratio test is used in normal distribution statistics to compare the variances s_1^2 and s_2^2 $(s_2^2 \leqslant s_1^2)$ of two samples the variances of which are based on m_1 and m_2 degrees of freedom. If the variances σ_1^2 and σ_2^2 of the two populations represented are not significantly distinct, then the ratio s_1^2/s_2^2 is given by

$$\frac{s_1^2}{s_2^2} \leq F_{m_1, m_2, \alpha}$$

where F is a standard statistic whose value has been tabulated for various pairs (m_1, m_2) and at different probability levels, α. Transposing this test into vector statistics gives, for $\delta_1^2 \geqslant \delta_2^2$,

(15) $$\frac{\hat{\delta}_1^2}{\hat{\delta}_2^2} \leq F_{2(N_1-1), 2(N_2-1), \alpha}$$

if two samples having respectively variances $\hat{\delta}_1^2$ and $\hat{\delta}_2^2$ and sizes N_1 and N_2 represent populations having approximately the same variances, δ^2. If however $\hat{\delta}_1^2/\hat{\delta}_2^2$ is greater than $F_{2(N_1-1), 2(N_2-1), \alpha}$, the result will be interpreted in the sense that $\hat{\delta}_1^2$, is significantly larger than $\hat{\delta}_2^2$ at the probability level α.

It will be pointed out here that Watson and Irving (1957) and later Irving (1964) proposed that the condition for two populations to be considered as having the same variance is that $\hat{\delta}_1^2$ be approximately equal to $\hat{\delta}_2^2$. This could easily be interpreted as an oversimplification of the test because, as it may be verified in the F-distribution tables, $\hat{\delta}_1^2/\hat{\delta}_2^2 = F = 1$ is only true when m_1 and m_2 (or $2(N_1 - 1)$ and $2(N_2 - 1)$ both tend toward infinity.

A second test based on variance estimate ratios may be used to determine whether or not the mean directions of a number of vectors are significantly distinct. Assuming that a sample of N unit vectors may be broken down into B groups of N_i ($i = 1, 2, 3, \ldots B$) vectors each and of resultant R_i, the angular variance estimate of the i^{th} group would be given by

$$\hat{\delta}_i^2 = \frac{2(N_i - R_i)}{(N_i - 1)}$$

Assuming that all groups have a common variance $\hat{\delta}_\omega^2$, we may also write

(16) $$(N_i - 1)\hat{\delta}_\omega^2 = 2(N_i - R_i)$$

and

(17) $$\hat{\delta}_\omega^2 = \frac{2\sum(N_i - R_i)}{\sum(N_i - 1)} = \frac{2(N - \sum R_i)}{(N - B)}$$

This last statistic is essentially a joint estimate of the within-site angular variance.

If we next assign the modulus $(\Sigma R_i / B)$ to each of the B group resultants we may assume that the modulus of the resultant R' of these normalized vectors is in practice almost equal to that of the group mean resultants R. On the basis of (11), the angular variance of these vectors is thus given by

$$(18) \qquad \hat{\delta}_b^2 = \frac{2\left(B\sum\left(\frac{R_i}{B}\right) - R\right)}{(B-1)} = \frac{2\left(\sum R_i - R\right)}{(B-1)}$$

Using the statistic F, as in the previous test, we may compare $\hat{\delta}_b^2$ and $\hat{\delta}_w^2$, remembering that the two variance estimates are based on 2(B - 1) and 2(N - B) degrees of freedom respectively. Thus, if

$$(19) \qquad \frac{\hat{\delta}_b^2}{\hat{\delta}_w^2} \leqslant F_{2(B-1),\,2(N-B),\,\alpha}$$

it may be concluded that the mean directions of the B groups are not significantly distinct at the probability level α and vice-versa. In cases where equation (19) is verified, the N samples may be considered as drawn from the same population if, furthermore, the ratio of the maxium $\hat{\delta}_i^2$ over the minimum $\hat{\delta}_i^2$ fulfils equation (15), i.e. satisfies the first test. If only equation (19) is satisfied, the B groups are representing several populations centred about the same mean direction.

It is interesting to study two limit cases which will allow us to establish the relationship between $\hat{\delta}^2$, $\hat{\delta}_b^2$ and $\hat{\delta}_w^2$. In the first case, it is assumed that the resultants of all B groups are parallel, i.e. that $\Sigma R_i = R$. Under such conditions,

$$(20) \qquad \hat{\delta}_b = 0 \qquad ; \qquad \hat{\delta}_w^2 = \frac{2(N-R)}{(N-B)} = \frac{\hat{\delta}_1^2(N-1)}{(N-B)}$$

where $\hat{\delta}_1^2$ is the angular variance estimate of such a sample. Assuming another sample of the same size but for which $\Sigma R_i = N$, yields

$$(21) \qquad \hat{\delta}_w^2 = 0 \quad ; \quad \hat{\delta}_b^2 = \frac{2(N-R)}{(B-1)} = \frac{\hat{\delta}_2^2(N-1)}{(B-1)}$$

As the combined dispersions of the two samples is given by

$$(22) \qquad \hat{\delta}^2 = \hat{\delta}_1^2 + \hat{\delta}_2^2 \quad ,$$

then

$$(23) \qquad \hat{\delta}^2 = \frac{1}{(N-1)}\left((B-1)\hat{\delta}_b^2 + (N-B)\hat{\delta}_w^2\right)$$

or, expressed in terms of "precision indexes",

$$(24) \qquad \frac{1}{k} = \frac{1}{(N-1)}\left(\frac{(B-1)}{k_b} + \frac{(N-B)}{k_w}\right)$$

Equation (24) differs substantially from that proposed by Watson and Irving (1957) who write

$$(25) \qquad k = \frac{1}{(k_w N)^{-1} + (k_b B)^{-1}}$$

The discrepancy lies primarily in the fact that neither k and k_b have the same meaning in the two equations. Watson and Irving define k_b in (25) by the relation

$$\frac{\Sigma R_i - R}{2(B-1)} = \frac{1}{2}\left(\frac{1}{k_w} + \frac{\bar{N}}{k_b}\right) \tag{26}$$

where $\bar{N}$ refers to a weighed average of N_i and is given by

$$\bar{N} = \frac{1}{(B-1)}\left(N - \frac{\Sigma N_i^2}{N}\right) \tag{27}$$

The compatibility of the last three equations with the above theory is not easily established but their validity and the physical meaning of k and k_b in them has no bearing on the validity of the test of significance described earlier.

The derivation of the statistical parameters in this paper is based on no particular density distribution such as those proposed by Fisher (1953) or Roberts and Ursell (1960). All that is required is that the individual vectors are drawn at random and that they satisfy the inequality

$$\frac{\delta_i^4}{4!} << 1 - \frac{\delta_i^2}{2} \tag{28}$$

TABLE I - SAMPLE DATA

SITE	OBSERVATION	DECLINATION (degrees)	INCLINATION (degrees)
A	1	12.6	46.8
"	2	9.5	46.9
"	3	6.6	39.0
"	4	7.8	33.7
"	5	7.7	40.9
"	6	1.5	38.8
B	1	2.0	34.0
"	2	11.5	60.3
"	3	5.9	37.5
"	4	2.6	35.9
"	5	8.0	35.0

TABLE II - STATISTICAL PARAMETERS

SAMPLE	MEAN DECLINATION (degrees)	MEAN INCLINATION (degrees)	N	R	$\hat{\delta}^2$ (radians)^2	R_o (.01)
Site A	7.47	41.06	6	5.97447	.02087	4.480
Site B	5.52	40.51	5	4.92062	.03969	4.023
Sites A + B	6.49	40.79	2	1.99981	.00038	-
11 vectors	6.59	40.82	11	10.89407	.02118	6.252

which is true when $\delta_i \leq 1$ radian and is very common with palaeomagnetic vectors. The valid application of the tests presupposes however that the individual vectors represent independent observations made at random on a formation supposedly polarized magnetically according to the same laws of hazard which govern the Gaussian distribution in scalar statistics. Since the individual directions of magnetization of the several specimens from a sample were not oriented independently, it is clear that it would be a misuse of the tests to consider these directions as independent observations in applying those tests.

It will be mentioned that these tests were all derived by Watson (1956-a) on a rigid mathematical basis for the particular case where the vectors are distributed according to Fisher's (1953) density function.

NUMERICAL EXAMPLE

The manipulation of the above tests is very simple, as will be illustrated by their application to a sample of palaeomagnetic directions already published elsewhere (Larochelle and Wanless, 1966). Eleven independently oriented samples were collected at two widely separated sites and their directions of magnetization were measured after alternating field cleaning treatment at 350 oersteds. From the directions of the individual vectors which are listed in Table I, we may calculate the mean direction at either site, the mean of the site mean directions and that of the 11 vectors. These directions are listed in Table II in terms of their declinations and inclinations with respect to the astronomic North and the horizontal respectively. In the same table the corresponding values of $\hat{\delta}^2$ were calculated on the basis of equation (11) and the values of R_o for N equal to 5, 6 and 11 at the probability level of .01 were read off a table published by Stephens (1964).

A comparison of R_o and R on each line of Table II indicates clearly that the probability of the formation studied being magnetized isotropically is significantly less then .01.

It may also be verified that the ratio $\hat{\delta}^2_B / \hat{\delta}^2_A$ (3.886) is slightly greater than $F_{8,\ 10,\ .05}(3.0717)$ but smaller than $F_{8,\ 10,\ .01}(5.0567)$. This indicates that there is approximately a 3 per cent chance that the palaeomagnetic vectors obtained from site A and site B respectively were drawn from populations having identical dispersions. The difference could possibly be explained by petrographic conditions at each site and probably not by sampling errors.

The two statistics $\hat{\delta}^2_b$ and $\hat{\delta}^2_w$ were computed with the aid of equations (17) and (18). Their ratio,

$$\frac{\hat{\delta}_b^2}{\hat{\delta}_w^2} = \frac{.00245}{.02333} = 0.1$$

is considerably smaller than $F_{2,\ 18,\ .05}(3.5546)$. This result confirms that there is practically no between-site dispersion in the sample considered i.e. that the two site mean directions are not significantly distinct. It may also be verified that equation (23) is satisfied by the calculated values of $\hat{\delta}^2$, $\hat{\delta}_b^2$ and $\hat{\delta}_w^2$.

Because the two mean site directions are not significantly distinct, it appears that the formation studied (a diabase dyke) acquired its magnetization at the two sites within a relatively short period. The question of which of the mean of the two site mean directions or the mean of the 11 palaeomagnetic vectors should be used in estimating a pole position does not arise in the present case. The pole position estimated from either one of these mean directions and on the basis of an axially centred dipole earth's field is certainly a virtual geomagnetic pole and the information available from the sample cannot be used to study the secular variation at the time the rock cooled below its Curie point.

The standard angular error of the mean of the eleven observations which is given by equation (13) may be calculated from the data listed in Table II. Expressed in terms of degrees, its calculated value is 2.5°. Twice this angle represents the half-angle of the cone axed on the sample mean and within which the formation mean direction of magnetization lies with a probability of .98. This corresponds very closely as expected, with the radius $\alpha_{.95}$ of the circle of confidence (4.7°), as defined by Fisher (1953) when his density distribution is assumed.

REFERENCES

Fisher, R.A.
1953: Dispersion on a sphere; Proc. Roy. Soc. Ser. A, vol. 217, p. 295.

Irving, E.
1964: Paleomagnetism and its application to geological and geophysical problems; John Wiley and Sons, N.Y., London.

Kemptone, O.
1951: The design and analysis of experiments; John Wiley and Sons Inc., London.

Larochelle, A. and Wanless, R. K.
1966: The paleomagnetism of a Triassic diabase dike in Nova Scotia; J. Geophy. Res., vol. 71, No. 20, p. 4949.

Rayleigh, S. J. W.
1919: On the problem of random vibrations and random flights in one, two and three dimensions; Phil. Mag., vol. 37, p. 321.

Roberts, P. H. and Ursell, H. D.
1960: Random Walk on a sphere and on a Riemannian Manifold, Phil. Trans. Roy. Soc., London, Ser. A, vol. 252, p. 317.

Stephens, M. A.
1964: The testing of unit vectors for randomness; J. Am. Statistical Assoc., vol. 59, p. 160.

Vincenz, S. A. and Bruckshaw, J. M.
1960: Note on the probability distribution of a small number of vectors; Proc. Cambridge Phil. Soc., vol. 56, pt. 1, p. 21.

Watson, G. S.
1956a: Analysis of dispersion on a sphere; Monthly Notices, Roy. Astron. Soc; Geoph. Suppl., vol. 7, No. 4, p. 153.

Watson, G. S.
1956b: A test of randomness of directions; Monthly Notices, Roy. Astron. Soc.; Geoph. Suppl., vol. 7, No. 4, p. 160.

Watson, G. S. and Irving, E.
1957: Statistical Methods in Rock Magnetism; Monthly Notices Roy. Astron. Soc.; Geoph. Suppl. vol. 7, No. 6, p. 289.

Watson, G. S. and Williams, E. J.
1956: On the construction of significance tests on the circle and the sphere; Biometrika, vol. 43, p. 344.