GEOLOGICAL SURVEY OF CANADA
COMMISSION GÉOLOGIQUE DU CANADA

This document was produced by scanning the original publication.

Ce document est le produit d'une numérisation par balayage de la publication originale.

PAPER 78-25

CONFIDENCE LIMITS IN PALEOMAGNETISM: ALTERNATIVES TO FISHER'S SOLUTION

A. LAROCHELLE

Energy, Mines and Resources Canada
Énergie, Mines et Ressources Canada

1979

GEOLOGICAL SURVEY
PAPER 78-25

CONFIDENCE LIMITS IN PALEOMAGNETISM: ALTERNATIVES TO FISHER'S SOLUTION

A. LAROCHELLE

1979

©Minister of Supply and Services Canada 1979

Available in Canada through

authorized bookstore agents
and other bookstores

or by mail from

Canadian Government Publishing Centre
Supply and Services Canada
Hull, Québec, Canada K1A 0S9

and from

Geological Survey of Canada
601 Booth Street
Ottawa, Canada K1A 0E8

A deposit copy of this publication is also available
for reference in public libraries across Canada

Cat. No. M44-78/25 Canada: $2.50
ISBN – 0-660-10192-0 Other countries: $3.00

Price subject to change without notice

Critical readers

F.P. Agterberg
J. Conaway

Original manuscript submitted: 1978 - 9 - 8
Approved for publication: 1978 - 9 - 26

CONTENTS

Illustrations

Tables

CONFIDENCE LIMITS IN PALEOMAGNETISM: ALTERNATIVES TO FISHER'S SOLUTION

Abstract

On the basis of direct analogies with the Gaussian probability density function, an expression is derived to define the density of unit vectors about the mean of a normally distributed population. Given a sample of N vectors drawn from such a population, an expression is derived to estimate the population angular standard deviation and the sample vector resultant is shown to be the best indicator of the population mean direction.

In addition to a solution proposed by Fisher in 1953 to determine the confidence limit about the sample resultant, two alternate solutions are described. The first of these may be expressed as:

$$\alpha_{95} \simeq \frac{\sqrt{3}\,R}{(R-1)} \frac{\hat{\delta}}{\sqrt{R}} \quad , \quad R > 2$$

where R is the length of the sample vector resultant and $\hat{\delta}$ *is the estimate of the population angular standard deviation. This solution is shown to yield values of* α_{95} *less than 0.5 degree larger than the values yielded by Fisher's solution when* $\hat{\delta}$ *is less than 20 . The second and more exact solution may be expressed as:*

$$\alpha_{95} = \sqrt{F_{2,2(N-1),\ .05}}\ \frac{\hat{\delta}}{\sqrt{R}}$$

where $F_{2,2(N-1),\ .05}$ *refers to the tabulated value of the F distribution. This solution appears to yield more accurate values of* α_{95} *than Fisher's solution, especially for small values of N and large values of* $\hat{\delta}$.

T The analysis is extended to the description of a method for establishing the most reliable indicator of a unit vector population mean direction when the N vectors relate to B distinct samples. The appropriate procedure for calculating α_{95} *in such cases is described.*

Résumé

A partir d'analogies directes avec la distribution de densité de Gauss, on dérive une expression qui définit la densité des vecteurs unitaires autour de la direction moyenne d'une population normale. En supposant un échantillon de N vecteurs tirés au hasard d'une telle population, on dérive une expression pour estimer la valeur de l'écart-type angulaire de la population et on démontre que la résultante vectorielle de l'échantillon constitue la meilleure indication de la direction moyenne de la population.

En plus de la formule proposée par Fisher (1953) pour déterminer la limite de confiance autour de la résultante de l'échantillon, deux solutions inédites sont données. La première peut s'exprimer par:

$$\alpha_{95} \simeq \frac{\sqrt{3}\,R}{(R-1)} \frac{\hat{\delta}}{\sqrt{R}}, \quad R > 2$$

où R désigne le module de la résultante vectorielle de l'échantillon et $\hat{\delta}$ *désigne l'estimation de l'écart-type angulaire de la population. Cette solution fournit des valeurs de* α_{95} *supérieures aux valeurs données par la solution de Fisher par moins de 0.5 quand la valeur de* $\hat{\delta}$ *est inférieure à 20 . La seconde solution, plus exacte, peut être exprimée par:*

$$\alpha_{95} = \sqrt{F_{2,2(N-1),\ .05}}\ \frac{\hat{\delta}}{\sqrt{R}}$$

où $F_{2,2(N-1),\ .05}$ *désigne la valeur de la distribution F tirée des tables de statistiques. Cette solution s'avère plus précise que la solution de Fisher, surtout si N est petit et* $\hat{\delta}$ *est élevé.*

L'analyse s'étend à la description d'une méthode pour établir la meilleure indication de la direction moyenne d'une population de vecteurs unitaires lorsque les N vecteurs qui la représentent sont répartis en B échantillons indépendants. La méthode appropriée pour la determination de α_{95} *dans ces cas est décrite.*

CONFIDENCE LIMITS IN PALEOMAGNETISM: ALTERNATIVES TO FISHER'S SOLUTION

INTRODUCTION

In most sectors of experimental research, statistical analysis is constantly used to quantify the reliability of scientific inferences whenever these are based on repeated observations or measurements of a parameter which appear to fluctuate haphazardly about a central tendency. In the field of paleomagnetism, the usefulness of statistics was widely recognized a quarter of a century ago, as evidenced by the sudden surge of statistical terminology in the literature of that time, shortly after a classical paper on the subject was written by Sir Ronald Fisher (1953).

It had been noted many decades earlier that the in situ remanent magnetization of a given rock unit often displays the tendency to line up more or less coherently along a mean direction which was associated with the local earth's magnetic field direction at the time the rock first consolidated. It was cautioned, however, that the rock unit as a whole could have acquired its remanent magnetization over a period of time long enough to span appreciable variations of the local geomagnetic field orientation. The possibility of this having happened provided a convenient explanation for the wide directional scattering often observed in sets of magnetization results, at a time when suitable sampling and measuring techniques as well as magnetic cleaning devices were not yet fully developed.

There are, of course, many other factors which may contribute to remanent magnetization directional scattering within a rock unit. One could identify lightning, local paleomagnetic field anomalies, chemical alterations relating to nearby intrusions or to weathering, etc. The net result of past tectonic activity may often be measured in the field and taken into account in estimating the mean direction of magnetization and the angular dispersion index of a rock formation. On the other hand, attempts to single out a correction for any of the other scattering factors enumerated above would be difficult if not totally speculative. At best, one may try to evaluate their combined influence and decide whether or not the rock unit under consideration still bears a reliable record of the ancient magnetic field direction, despite any past activity of the various scattering factors. Statistical analysis can provide a useful guide in making such a decision.

Measures of in situ directions of magnetization are further subject to angular dispersion through experimental errors introduced during the sampling and measuring operations. Although these errors may be largely eliminated by careful manipulation and controls, statistical analysis can help here again in the evaluation of the relative importance of this angular scattering factor.

Most papers reporting paleomagnetic data at present include an index "k" quantifying the estimated degree of clustering among the magnetization directions within a rock unit. Papers also generally state the limit of confidence "α_{95}" about the vector resultant of a sample within which limit the central tendency of the population under study is likely to be located, at the 0.95 probability level.

It is obvious that statistics may be utilized much beyond that extent, be it to justify the merging of independent sets of data, to differentiate paleomagnetic subpopulations within a set of data, to test the likely compliance of a set of data with a theoretical distribution, to flag inadequately sampled rock units, to evaluate the probability of random distribution in widely scattered data or, in general, to assess the credibility of a stated paleomagnetic interpretation. Most of these applications were suggested some years ago by Watson (1956) and Watson and Irving (1957), although there are relatively few of them reported in the current literature.

The specific aspect of statistical analysis dealt with in the present paper relates to the method of calculating the limit of confidence α_{95} referred to above. Fisher (1953) proposed such a method many years ago and its early acceptance by paleomagnetists is clear evidence that it fulfilled the needs then and still does, most adequately. Nevertheless, many a paleomagnetist will admit even today that he has not read Fisher's paper or that he has failed to follow its mathematical derivations. The purpose of the present paper is to illustrate, in terms perhaps better adapted to the nonmathematician, that the implications of Fisher's solution can be arrived at through a somewhat different path.

Acknowledgment

The writer wishes to acknowledge J.G. Conaway's many useful suggestions to improve on the clarity of the original manuscript.

ANGULAR DISTRIBUTION OF A PALEOMAGNETIC POPULATION

A paleomagnetic population may be pictured as a bundle of vectors radiating from a common origin which coincides with the centre of a unit radius spherical shell. Assuming that the population mean direction intersects the sphere at a point "P", let us further denote by "α_i" the arc of circle joining the intersection of the i^{th} vector with the spherical shell and point P. For a normal distribution, the dispersion is such that the population vectors intersect the spherical shell symmetrically about point P and the density of these intersections on a spherical zone of width $\Delta\alpha$, radius $\sin\alpha$, and centre P, would decrease as α increases. For a given value of α, therefore, the density of the intersections on the spherical zone is dependent on the characteristic angular cohesion of the population under study, as this reflects the combined effectiveness of the scattering factors enumerated above. Thus the density of the intersections around point P may be expressed as a function of α and of the angular cohesion of the population.

The characteristic angular cohesion of a population may be defined quantitatively in terms of the population's angular standard deviation "δ", the counterpart of σ in the Gaussian distribution. Thus the density of the intersections around point P may be expressed as $f(\alpha, \delta)$.

With reference to Figure 1 and to the fact that δ is constant for a given population, we may write that the probability for the spherical zone of radius $\sin\alpha_i$ to be intersected by a vector singled out at random in the population is given by:

$$\Delta P \left[\left(\alpha_i - \frac{\Delta\alpha}{2}\right) \leqslant \alpha_i \leqslant \left(\alpha_i + \frac{\Delta\alpha}{2}\right) \right] = \frac{2\pi \sin\alpha_i \, \Delta\alpha \, f(\alpha_i, \delta)}{4\pi} \quad (1)$$

By direct analogy with the Gaussian probability density function, which may be expressed as:

$$f(\mu - x, \sigma) = \Phi(\sigma) e^{-(\mu - x)^2 / 2\sigma^2} \quad (2)$$

it is suggested that $f(\alpha, \delta)$ is of the form:

$$f(\alpha, \delta) = \Psi(\delta) e^{-h\alpha^2/\delta^2} \quad (3)$$

where α and δ fulfill respectively the roles of $(\mu - x)$ and σ in equation (2). The parameter h in equation (3) is a constant (like 1/2 in equation (2)) which will be determined through further analogy with the Gaussian system, but let us first define explicitly $\Psi(\delta)$, the counterpart of $\Phi(\sigma)$ in equation (2). It is known that

$$\Phi(\sigma) = 1/\sqrt{2\pi}\;\sigma$$

For values of δ smaller than $\pi/8$, it can be shown that the relation $\alpha^2 = 2(1-\cos\alpha)$ is valid for most vectors in the population, so that equation (3) may be rewritten as:

$$f(\alpha, \delta) = \frac{\Psi(\delta) e^{2h\cos\alpha/\delta^2}}{e^{2h/\delta^2}} \quad (4)$$

Since a vector singled out at random in the population intersects the spherical shell at one point or another, by combining equations (1) and (4), we may write the following identity:

$$P\left[\alpha \leqslant \pi\right] = \int_{-1}^{1} \frac{\Psi(\delta) e^{2h\cos\alpha/\delta^2} d(\cos\alpha)}{2e^{2h/\delta^2}} \equiv 1 \quad (5)$$

which yields:

$$\Psi(\delta) = \frac{2he^{2h/\delta^2}}{\delta^2 \sinh(2h/\delta^2)} \quad (6)$$

It follows that a vector singled out at random in the population deviates from the population mean direction by an angle $\alpha \leqslant \alpha_o$, with probability P, given by:

$$P\left[\alpha \leqslant \alpha_o\right] = \int_{\cos\alpha_o}^{1} \frac{he^{2h\cos\alpha/\delta^2} d(\cos\alpha)}{\delta^2 \sinh(2h/\delta^2)} \quad (7)$$

Noting that for values of h greater than 0.75 and for values of δ smaller than $\pi/8$,*

$$2\sinh(2h/\delta^2) = e^{2h/\delta^2}$$

and we may write the last equation as:

$$P\left[\alpha \leqslant \alpha_o\right] = 1 - e^{-h\alpha_o^2/\delta^2} \quad (8)$$

$*\sinh(ax) = \frac{e^{ax} - e^{-ax}}{2}$

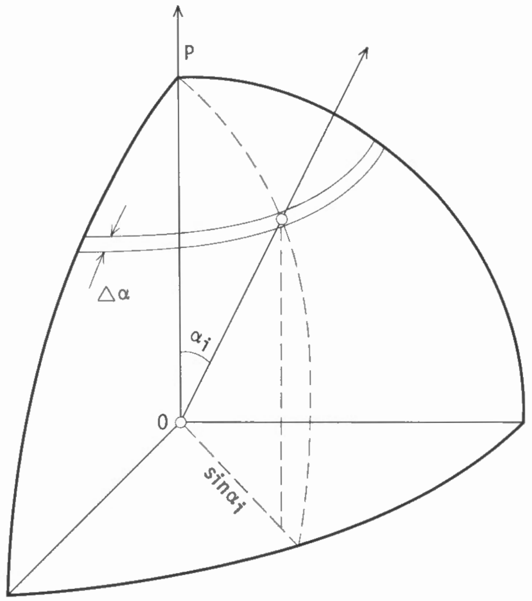

Figure 1. Paleomagnetic vector intersecting spherical zone of radius $\sin\alpha$ *and width* $\Delta\alpha$.

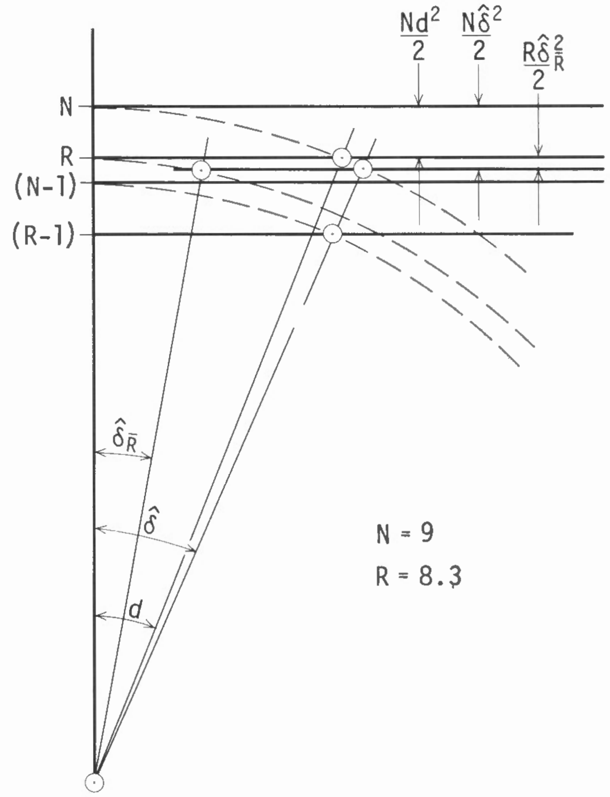

Figure 2. Graphical method of determining d, $\hat{\delta}$, *and* $\hat{\delta}_{\bar{R}}$ *for a sample of given N and R.*

Of particular interest are the values of $P[\alpha \leqslant \delta]$ and $P[\alpha \leqslant 2\delta]$ for discrete values of h. Table 1 lists some of these values and shows that:

$$P\left[\alpha \leqslant \delta\right] = P\left[(\mu-\sigma) \leqslant x \leqslant (\mu+\sigma)\right] = 0.6826$$

when h = 1.1476, and

$$P\left[\alpha \leqslant 2\delta\right] = P\left[(\mu-2\sigma) \leqslant x \leqslant (\mu+2\sigma)\right] = 0.9546$$

when h = 0.7731. It is suggested then that the value of h, which would make equation (7) most compatible with the Gaussian system, lies in the interval between 0.7731 and 1.1476. Within that interval, the choice of h=1 is an advantageous one as it simplifies the writing of equation (7) and makes its handling less cumbersome than would a fractional number. This choice has the additional advantage of leading precisely to the probability density function universally adopted in paleomagnetic research, i.e.:

$$P\left[\alpha \leqslant \alpha_o\right] = \int_{\cos\alpha_o}^{1} \frac{\kappa e^{\kappa\cos\alpha} d(\cos\alpha)}{2\sinh(\kappa)} \tag{9}$$

where κ actually fulfills the role of $2/\delta^2$ in equation (7). Equation (9) is generally attributed to Fisher (1953) although it was utilized earlier by Arnold (1941) and first appeared in the statistical mechanics literature in a paper by Langevin (1905).

Having set h=1 in equation (8), we may then derive from that basis the value of "r" in the expression:

$$P\left[\alpha \leqslant r\delta\right] = 0.95 \tag{10}$$

It will be recognized that $r\delta$ is the half angle of the circular cone coaxial with the population mean direction and containing 95 per cent of the vectors in the population. It may be verified that

$$r = 1.7308 \simeq 1.7320 = \sqrt{3}$$

Equation (8) may also be used to show that 63.21 per cent of the vectors in a population lie within one angular standard deviation, δ, from the population mean direction, i.e. that:

$$P\left[\alpha \leqslant \delta\right] = 0.6321$$

POPULATION MEAN DIRECTION "BEST" INDICATOR

Determining the "best" indicator of the mean direction of a unit vector population on the basis of a sample drawn at random from it is an operation which is analogous to fitting the "best" straight line through a set of points assumed to be related by a hypothetical linear function. In both cases the approach is the same and consists in determining, on the basis of the available sample, the equation of the line (direction in space) which deviates least from the individual points (unit vectors) in the sample. The sum of the individual deviations (angular deviations) squared is minimal for that line (direction in space) which is considered the "best" straight line (the "best" indicator of the population mean direction).

Assume a sample of N unit vectors drawn at random from a common population. Denoting by γ_i the angular deviation of the i^{th} vector from any referential axis in space, the sum of the squared angular deviations from that axis, $\Sigma\, \gamma_i^2$, assumes a minimum value when $\Sigma \cos\gamma_i$ is maximal. As

$$\Sigma \cos\gamma_i = N - \Sigma\, \gamma_i^2/2$$

assumes a maximum value when γ_i relates the i^{th} vector to the resultant R of the N vectors, it follows that the direction cosines of R coincide with those of the "best" indicator of the population mean direction, as far as the information contained in the sample can indicate. This may be illustrated by the simple example of two unit vectors deviating from each other by an angle ϕ. Assuming any coplanar axis through the origin of the two vectors and deviating by an angle β from their resultant R, the sum of the deviations squared is given by:

$$\Sigma\gamma_i^2 = (\beta-\phi/2)^2 + (\beta+\phi/2)^2 = 2\beta^2 + \phi^2/2$$

and is minimal when $\beta=0$, i.e. when the axis coincides with R.

POPULATION ANGULAR DISPERSION INDEX

Denoting by Θ_i the angular deviation between the i^{th} vector and R in the unit vector sample described in the previous section, and assuming that in this context

$$\Theta_i^2 = 2(1-\cos\Theta_i)$$

we may write:

$$R = \Sigma\cos\Theta_i = N - \frac{\Sigma\Theta_i^2}{2} \qquad (11)$$

and define

$$d^2 = \frac{\Sigma\Theta_i^2}{N} = \frac{2(N-R)}{N} \qquad (12)$$

as the <u>sample angular variance</u>, corresponding to s^2 in the Gaussian system.

The value of d^2 clearly reflects the degree of angular dispersion in the population but it must be considered generally as a biased estimate of the <u>population angular variance</u>, δ^2, since its calculation is based on the sum of the squared angular deviations from the sample mean rather than from the actual population mean direction. Thus, the larger the angular deviation between R and the population mean direction, the less reliable is d^2 as an estimate of δ^2 and this is most likely to occur when the sample size N is small and/or when δ^2 is large. The bias in d^2 may then be expected to be directly proportional to δ^2 and inversely proportional to N. Denoting by $\hat{\delta}^2$ <u>the corrected estimate of δ^2</u>, we may write:

$$\hat{\delta}^2 = d^2 + \frac{\hat{\delta}^2}{N} = \frac{2(N-R)}{(N-1)} \qquad (13)$$

The angular variance estimate $\hat{\delta}^2$ may be considered as the counterpart of $\hat{\sigma}^2$ in the Gaussian system, the unbiased estimate of σ^2.

Using again the relation $d^2 = 2(1-\cos d)$, equations (12) and (13) may be rewritten respectively as:

$$\cos d = R/N \qquad (12b)$$

and

$$\cos\hat{\delta} = \frac{R-1}{N-1} \qquad (13b)$$

and use these two equations to illustrate the geometric significance of d and $\hat{\delta}$, as shown in Figure 2 for a hypothetical sample of size N=9 and resultant R=8.3.

Figure 2 also provides the opportunity of introducing $\delta_{\bar{R}}$, the counterpart of $\sigma_{\bar{x}}$ in the Gaussian system, referred to as the standard deviation (or error) of the sample mean. The significance of $\delta_{\bar{R}}$ is explained as follows.

Suppose we consider a population of 100 unit vectors having an angular standard deviation δ and a mean direction defined by direction cosines (λ, μ, ν). Taking all possible samples of size N=10 from that population (there are 100!/(90! 10!) or more than 10^{13} of them!) and recording the mean direction of each would give us a population of sample mean directions. That population would have a mean direction defined by direction cosines (l,m,n) and its angular standard deviation would be $\delta_{\bar{R}}$.

It is obvious that (l,m,n) would match exactly (λ, μ, ν) because the 100 vectors in the original population would appear an equal number of times in the making of the 10^{13} samples of size N=10.

On the other hand, $\delta_{\bar{R}}$ may reasonably be expected to be smaller than δ, as the mean direction of each of the samples of size N=10 is a "best" estimate of the population mean direction and is thus closer to it than the "average" vector in that sample. This implies that the population of sample means is more tightly distributed than the original 100-vector population, and thus that $\delta_{\bar{R}} < \delta$. Since, as illustrated in Figure 2,

$$N\hat{\delta}^2 = Nd^2 + R\hat{\delta}_{\bar{R}}^2, \qquad (14)$$

equation (13) may be used to show that:

$$\hat{\delta}_{\bar{R}}^2 = \hat{\delta}^2/R \qquad (15)$$

i.e.

$$\hat{\delta}_{\bar{R}} = \hat{\delta}/\sqrt{R}$$

and

$$\cos\hat{\delta}_{\bar{R}} = \frac{N(R-1)}{R(N-1)} \qquad (16)$$

LIMIT OF CONFIDENCE ABOUT THE SAMPLE RESULTANT

Having determined that the sample resultant R has the direction of the "best" indicator of the population mean tendency, it is relevant to evaluate how reliable that indicator is, i.e. to determine the maximum angle the population mean direction could be expected to deviate from R, at a stated probability level. The limit of confidence about R may be visualized as a circular cone coaxial with R and the apex of which coincides with the centre of the unit spherical shell mentioned earlier. The intersection of that cone with the spherical shell would appear on a stereographic projection as a circle, hence the expression of the radius of the circle of confidence to define the half-angle of the cone of confidence. The probability level generally adopted in practice is 0.95 and α_{95} is the symbol most commonly used to designate the limit of confidence about the sample resultant.

Assuming that δ^2 is known for a given population, it is easy to derive a formula to calculate the value of α_{95} relating to the resultant of a sample drawn from that population. Referring back to equation (7) (where h=1), we may express the frequency element of $\cos\alpha$ for the population as:

$$dF(\cos\alpha) = \frac{e^{2\cos\alpha/\delta^2}\, d(\cos\alpha)}{\delta^2 \sinh(2/\delta^2)} \qquad (17)$$

Similarly, considering the sample mean direction as a unit vector drawn at random from the population of sample mean directions, we may express the frequency element of $\cos\alpha$ for that population as:

$$dF(\cos\alpha) = \frac{e^{2\cos\alpha/\delta_R^2}\, d(\cos\alpha)}{\delta_R^2 \sinh(2/\delta_R^2)} = \frac{Re^{2R\cos\alpha/\delta^2}\, d(\cos\alpha)}{\delta^2 \sinh(2R/\delta^2)} \qquad (18)$$

and, since in this context,

$$\sinh(2R/\delta^2) = e^{2R/\delta^2}/2$$

it follows that:

$$dF(\cos\alpha) = \frac{2Re^{-2R(1-\cos\alpha)/\delta^2}\, d(\cos\alpha)}{\delta^2} \qquad (19)$$

Integrating the last equation between the limits $\cos\alpha_o$ and 1 yields the probability $P[\alpha \leq \alpha_o]$ that the population mean direction may deviate from the sample mean direction by an angle equal to or smaller than α_o. Conversely, given the probability level P and integrating equation (19) between the limits $\cos\alpha_p$ and 1 yields the maximum deviation expected between the population and the sample mean directions, at the chosen probability level P. In particular,

$$P\left[\alpha \leqslant \alpha_{95}\right] = 0.95 = \int_{\cos\alpha_{95}}^{1} \frac{2Re^{-2R(1-\cos\alpha)/\delta^2}\, d(\cos\alpha)}{\delta^2} \qquad (20)$$

or

$$\alpha_{95} = 1.7308\ \delta/\sqrt{R} \simeq \delta\sqrt{3/R} = \sqrt{3}\ \ \delta_R \qquad (21)$$

Therefore, if δ is known for a given population, either as a result of extensive sampling or otherwise, the limit of confidence about the mean direction of a sample drawn from it at random can readily be determined with the aid of equation (21).

In most cases, however, δ is not known and only an estimate of this statistic, $\hat{\delta}$, is available. It would not be safe to replace δ by $\hat{\delta}$ in equation (21) for the determination of α_{95} because the range of values which $\hat{\delta}$ could assume is relatively wide, especially if the size of the sample is very small and/or if δ is relatively large. Such an unreliable basis for the determination of a limit of confidence would hardly be satisfactory, and it is necessary to multiply $\hat{\delta}$ by a "safety factor" prior to incorporating it in equation (21). We shall denote this factor by T_N, the subscript referring to the sample size.

In order to derive a general expression for T_N, let us first consider the simple case of a sample consisting of two vectors which deviate from the population mean direction by angles α_1 and α_2 respectively. The probability of drawing two such vectors from the population follows from equations (1), (4), and (6) and may be expressed as:

$$P\left[\alpha_1,\alpha_2\right] = \frac{e^{2(\cos\alpha_1+\cos\alpha_2)/\delta^2} \sin\alpha_1 \sin\alpha_2\, d\alpha_1\, d\alpha_2}{\left[\delta^2 \sinh(2/\delta^2)\right]^2} \qquad (22)$$

so that the frequency element of the $(\cos\alpha_1+\cos\alpha_2)$ distribution is given by:

$$P\left[\alpha_1, \alpha_2\right] d\,(\cos\alpha_1 + \cos\alpha_2)$$

Given α_1 and α_2, the length of the two vectors' resultant, R, may be anywhere between $2\cos\phi_1$ and $2\cos\phi_2$, where

$$\phi_1 = (\alpha_1+\alpha_2)/2 \text{ and } \phi_2 = (\alpha_1-\alpha_2)/2$$

and the probability element of R may then be expressed as:

$$P\left[R\right] = dR\Big/\int_{2\cos\phi_1}^{2\cos\phi_2} dR = \frac{\cos(\alpha_1/2)\cos(\alpha_2/2)dR}{\sin\alpha_1 \sin\alpha_2} \qquad (23)$$

Furthermore, α being the angular deviation between R and the population mean direction, we may write:

$$\cos\alpha_1 + \cos\alpha_2 = R\cos\alpha$$

and

$$d(\cos\alpha_1 + \cos\alpha_2) = Rd(\cos\alpha)$$

so that the joint probability element of α_1, α_2, R, and $\cos\alpha$ may be expressed as:

$$dF(\alpha_1,\alpha_2,R,\cos\alpha) = \frac{e^{2R\cos\alpha/\delta^2}\cos(\alpha_1/2)\cos(\alpha_2/2)d\alpha_1 d\alpha_2 RdRd(\cos\alpha)}{[\delta^2\sinh(2/\delta^2)]^2} \quad (24)$$

Integrating this equation successively with respect to α_1 and α_2, both times within the limits of 0 and π, yields:

$$dF(R,\cos\alpha) = \frac{4e^{2R\cos\alpha/\delta^2}RdRd(\cos\alpha)}{[\delta^2\sinh(2/\delta^2)]^2} \quad (25)$$

The last equation may be integrated again with respect to $\cos\alpha$ within the limits -1 and +1, to give the marginal distribution of R:

$$dF(R) = \frac{4\delta^2\sinh(2R/\delta^2)dR}{[\delta^2\sinh(2/\delta^2)]^2} \quad (26)$$

Dividing equation (25) by equation (26) yields the frequency element of $\cos\alpha$, conditional to R being given:

$$dF(\cos\alpha) = \frac{Re^{2R\cos\alpha/\delta^2}d(\cos\alpha)}{\delta^2\sinh(2R/\delta^2)} \quad (27)$$

It is noted that the equivalent result was obtained through a different reasoning in equation (18).

Since

$$2\sinh(2R/\delta^2) = e^{2R/\delta^2}$$

in the present context, we may rewrite equations (26) and (27) respectively as:

$$dF(R) = \frac{2e^{-2(2-R)/\delta^2}dR}{\delta^2} \quad (28)$$

and

$$dF(\cos\alpha) = \frac{2Re^{-2R(1-\cos\alpha)/\delta^2}d(\cos\alpha)}{\delta^2} \quad (29)$$

Integrating equation (28) over the range R_o to 2 yields the probability of R being larger than R_o, i.e.:

$$P\left[R \geqslant R_o\right] = 1 - e^{-2(2-R_o)/\delta^2} \quad (30)$$

To eliminate δ^2 in the last equation, we may replace R_o by R and take the partial derivative of P with respect to $(2/\delta^2)$, i.e.

$$\frac{\partial P}{\partial(2/\delta^2)} d(2/\delta^2) = (2 - R)e^{-2(2-R)/\delta^2}d(2/\delta^2) \quad (31)$$

and multiply this frequency element by the frequency element of $\cos\alpha$. Integrating this product with respect to $(2/\delta^2)$ over the range $0 \leqslant 2/\delta^2 \leqslant \infty$ yields:

$$dF(\cos\alpha) \simeq \frac{R(2-R)d(\cos\alpha)}{(2-R\cos\alpha)^2} \quad (32)$$

Table 1

Values of $P[\alpha \leqslant \delta]$ and $P[\alpha \leqslant 2\delta]$ for discrete values of h

h	0.75	0.7731	1.00	1.1476	1.20
$P[\alpha \leqslant \delta]$	0.5276	0.5384	0.6321	0.6826*	0.6988
$P[\alpha \leqslant 2\delta]$	0.9502	0.9546*	0.9817	0.9899	0.9918

*Examples cited in text.

Table 2

Calculated values of R, α_{95} and T_2 relating to a sample of size 2, for discrete values of $\hat{\delta}$

$\hat{\delta}$	1°	5°	10°	15°	20°
R	1.99985	1.99619	1.98477	1.96573	1.93908
α_{95}	3.06°	15.48°	31.32°	48.02°	66.23°
T_2	2.49	2.52	2.55	2.59	2.66

Table 3

Calculated values of $\sqrt{3}\ T_N$ for various sample sizes and discrete values of $\hat{\delta}$ compared with the theoretical values of Student "t", at probability level 0.05 and for 2(N-1) degrees of freedom. The values of "t" were drawn from standard statistical tables.

N \ $\hat{\delta}$	1°	5°	10°	15°	20°	"t"
2*	4.31	4.36	4.41	4.48	4.60	4.30
3	2.60	2.60	2.61	2.63	2.65	2.78
4	2.31	2.31	2.32	2.33	2.35	2.45
6	2.08	2.08	2.08	2.09	2.10	2.23
10	1.92	1.93	1.93	1.93	1.94	2.10
15	1.86	1.86	1.86	1.86	1.86	2.05
20	1.82	1.82	1.82	1.83	1.83	1.99

*The values of $\sqrt{3}\ T_2$ were obtained from the values of T_2 listed in Table 2.

This expression for the frequency element of $\cos\alpha$ is independent of δ^2, and makes it possible to calculate the probability that $\cos\alpha$ exceeds a chosen value, $\cos\alpha_o$, or that α is smaller than or equal to a chosen angle, α_o, without having to rely on an estimate of δ^2. Conversely, given the probability P, the value α_p may be determined, with the same advantage. In the particular case where P = 0.95, we may write:

$$P\left[\cos\alpha \geqslant \cos\alpha_{95}\right] = P\left[\alpha \leqslant \alpha_{95}\right] = 0.95 \simeq \int_{\cos\alpha_{95}}^{1} \frac{R(2-R)d(\cos\alpha)}{(2-R\cos\alpha)^2} \tag{33}$$

so that:

$$0.95 \simeq 1 - \frac{(2-R)}{(2-R\cos\alpha_{95})}$$

or

$$\cos\alpha_{95} \simeq \frac{2}{R}\left[1-10(2-R)\right] = \sqrt{3}\ T_2\hat{\delta}/\sqrt{R} \tag{34}$$

This result is identical to that derived by Fisher (1953) for the specific case of N=2*. Equation (34) may be used to calculate the value of T_2 for any given value of $\hat{\delta}$, thus providing an insight as to the order of magnitude and the mode of variation of T_N for samples of larger sizes.

Given any value of $\hat{\delta}$ in the range $0 \leqslant \hat{\delta} \leqslant \pi/8$, the corresponding value of R may be calculated with the aid of equation (13) for a sample of size 2. Inserting these values of R and $\hat{\delta}$ into equation (34) yields the corresponding values of α_{95} and T_2. Values of R, α_{95}, and T_2 calculated on that basis for discrete values of $\hat{\delta}$ are listed in Table 2.

It is noted that T_2 increases slowly as $\hat{\delta}$ increases (i.e. as R decreases) and thus it may reasonably be expected that T_N should follow the same trend for larger values of N.

A second attribute of T_N is that it may be expected to decrease asymptotically to unity when N (and therefore R) increases indefinitely, because the need for a "safety factor" in estimating the value of δ is "averaged" out with large samples. This suggests that T_3 should be smaller than T_2 and larger than T_4, and so on.

It is noted that the product $\sqrt{3}\ T_2$ in equation (34) fulfills essentially a function identical to Student "t" in Gaussian statistics. It may be verified that in effect the product of $\sqrt{3}$ by any one of the values of T_2 listed in Table 2 is nearly equal to that of the "t" distribution for 2(N-1) degrees of freedom, at the 0.05 probability level. This confirms the validity of the approximation given by equation (34) for N=2, and it implies that the probability that the sample mean direction and the population mean direction diverge from one another by an angle larger than α_{95} is only about 0.05. Stated in a different way, the probability is approximately 0.95 that the two mean directions are within α_{95} from one another. This suggests a third characteristic of T_N, i.e. that the product $\sqrt{3}\ T_N$ has approximately the distribution of Student "t" with 2(N-1) degrees of freedom, at the 0.05 probability level, $"t"_{.05,2(N-1)}$.

As indicated above, the value of T_N for any specific case relates to the value of R, and we may then identify T_N as a function of R which decreases asymptotically to one as N (and R) increases indefinitely and which increases slowly as R decreases (or $\hat{\delta}$ increases) for a fixed value of N. We may then write tentatively:

$$T_N = R/(R-1) \tag{35}$$

and test whether the product of $\sqrt{3}$ by the right hand member of the last equation has approximately the value of $"t"_{.05,2(N-1)}$.

We may note first that, under the definition of equation (35),

$$T_3 \leqslant 1.5 < T_2$$

since

$$2.49 \leqslant T_2 \leqslant 2.66$$

so that a first requirement would be fulfilled by equation (35) for samples of sizes larger than 2.

The product of $\sqrt{3}$ by the right hand member of equation (35) is listed in Table 3 for discrete values of $\hat{\delta}$ and N. These are compared with the appropriate values of "t" which were drawn from standard statistical tables.

---

*It is noted, however, that Fisher's derivation is of questionable validity because it hinges on the false postulate that: $d(\cos\alpha_1)d(\cos\alpha_2) = 1/2\ d(\cos\alpha_1+\cos\alpha_2)d(\cos\alpha_1-\cos\alpha_2)$.

Table 4

Values of α_{95} calculated for samples of various sizes and angular standard deviations, with the aid of equation (36) (upper) and of Fisher's equation (lower)

N \ $\hat{\delta}$	1°	5°	10°	15°	20°
2*	3.06°	15.48°	31.32°	48.02°	66.23°
3	1.50 1.51	7.52 7.62	15.15 15.34	23.03 23.25	31.29 31.46
4	1.15 1.14	5.79 5.68	11.66 11.42	17.70 17.30	24.02 23.38
6	0.85 0.83	4.25 4.14	8.56 8.33	12.99 12.61	17.61 17.04
10	0.61 0.60	3.05 2.99	6.14 6.01	9.31 9.09	12.60 12.29
15	0.48 0.47	2.40 2.36	4.83 4.75	7.32 7.20	9.91 9.73
20	0.41 0.40	2.04 2.02	4.11 4.06	6.23 6.15	8.43 8.31

*The values of α_{95} for N=2, reproduced from Table 2, were calculated with the use of equation (34), which is identical to Fisher's solution for this specific case.

Table 5

Values of α_{95} calculated for samples of various sizes and angular standard deviations, with the aid of equation (37) (upper) and of Fisher's equation (lower)

N \ $\hat{\delta}$	1°	5°	10°	15°	20°
2	3.08° 3.06	15.43° 15.48	30.94° 31.32	46.63° 48.02	62.60° 66.23
3	1.52 1.51	7.62 7.62	15.29 15.34	23.09 23.25	31.07 31.46
4	1.13 1.14	5.68 5.68	11.40 11.42	17.23 17.30	23.21 23.38
6	0.83 0.83	4.14 4.14	8.32 8.33	12.58 12.61	16.97 17.04
10	0.60 0.60	2.99 2.99	6.00 6.01	9.08 9.09	12.26 12.29
15	0.47 0.47	2.36 2.36	4.75 4.75	7.19 7.20	9.71 9.73
20	0.40 0.40	2.01 2.02	4.05 4.06	6.13 6.15	8.28 8.31

It is noted that in all cases the calculated values of $\sqrt{3}\ R/(R-1)$ are approximately equal to "t", thus supporting the validity of equation (35) as an approximative solution. We may then write:

$$\alpha_{95} = \sqrt{3}\ T_N \hat{\delta}/\sqrt{R} \simeq \sqrt{3R}\ \hat{\delta}/(R-1) \qquad (36)$$

Table 4 lists the values of α_{95} computed with the aid of equation (36) for discrete values of $\hat{\delta}$ in the range of 1 to 20° and for various sample sizes. These values are listed immediately above those obtained with the use of the more elaborate equation due to Fisher (1953)* for the corresponding values of N and R.

It is noted that, except when N=3, the values of α_{95} yielded by equation (36) are always slightly larger than their counterpart, so that the confidence limits obtained with the use of this equation are slightly more conservative than those obtained with the use of Fisher's solution. It is interesting to note, in any case, that the discrepancies between the two values never exceed a fraction of a degree within the range considered.

Equation (36), however, is obviously not appropriate to calculate the limits of confidence about the mean directions of samples of size N=2. In this particular case, equation (34) applies.

The relatively good correspondence between the results yielded by Fisher's solution and equation (36) respectively reflects the equivalence of the assumptions on which the two solutions are based. These relate to the general condition that $\hat{\delta} \leqslant \pi/8$ (or $k \geqslant 13$), beyond which the values of α_{95} obtained with either solution are less and less meaningful as $\hat{\delta}$ increases.

A totally different approach to determine α_{95} stems from the definition of the statistic χ^2. According to the theory of statistics, if $t_1, t_2, t_3 \ldots.. t_N$ are N independent standard deviates and we define

$$\chi^2 = \Sigma t_i^2,$$

then χ^2 has the distribution

$$f(\chi^2) = \frac{e^{-\chi^2/2}(\chi^2)^{\nu/2-1}}{2^{\nu/2}\,\Gamma(\nu/2)}$$

which is known as the χ^2 distribution with ν degrees of freedom, i.e. with the number of degrees of freedom on which Σt_i^2 is based.

Assuming a sample of N unit vectors drawn at random from a normal population, let us denote by Θ_i the angular deviation between the sample vector resultant, R, and the i^{th} vector in the sample. Since each vector is drawn at random, (N-1) values of Θ_i are independent (the N^{th} one being defined by R) and Θ_i/δ may be considered as a standard independent deviate. Since, furthermore, each vector is based on 2 degrees of freedom (i.e. 2 of its 3 direction cosines), we may write:

$$\chi_\nu^2 = \frac{2\Sigma\Theta_i^2}{\delta^2} \simeq \frac{2\cdot 2(N-R)}{\delta^2}$$

where $\nu=2(N-1)$.

If we denote by α the angle between R and the population mean direction and by Z, the projection of R on the population mean direction, we have:

$$Z = R\cos\alpha \simeq R(1-\alpha^2/2)$$

or

$$R\alpha^2 \simeq 2(R-Z)$$

and, by analogy,

$$\chi_2^2 = \frac{2R\alpha^2}{\delta^2} \simeq \frac{2\cdot 2(R-Z)}{\delta^2}$$

since α has 2 degrees of freedom.

It is shown in statistics textbooks that if u is a standard independent variable having the χ^2 distribution with m degrees of freedom and if v is another standard independent variable having the χ^2 distribution with n degrees of freedom, then the ratio

$$F = \frac{u/m}{v/n}$$

has the distribution of the random variable $F_{m,n,p}$ which has been tabulated for various probability levels, p. For example, if p=0.05, the probability that F is smaller than $F_{m,n,.05}$ is 0.95.

---

*Fisher's equation may be written as: $\alpha_{95} \simeq \cos^{-1}\left[1 - \frac{N-R}{R}(20^{1/(N-1)}-1)\right]$

Applying this theorem to the above context, we may then write:

$$F_{2,2(N-1),.05} = \frac{R\alpha_{95}^2(N-1)}{2(N-R)}$$

where α_{95} is the maximum value expected for α with a probability of 0.95. It follows from the last equation that:

$$\alpha_{95} = \sqrt{F_{2,2(N-1),.05}}\ \frac{\hat{\delta}}{\sqrt{R}} \qquad (37)$$

It may be verified that, for an infinitely large sample,

$$\alpha_{95} = 1.7308\ \frac{\hat{\delta}}{\sqrt{R}} \simeq \sqrt{3}\ \frac{\hat{\delta}}{\sqrt{R}}$$

as already indicated in equation (21).

Table 5 lists values of α_{95} calculated with the aid of equation (37) for discrete values of N and $\hat{\delta}$. It is noted that there is a close correspondence between these values and those obtained with the aid of Fisher's solution for the same values of $\hat{\delta}$ and N, except when N=2 and for the larger values of $\hat{\delta}$. In this specific case, the exact solution for α_{95} is given by:

$$F_{2,2,.05} = 19.000 = \frac{R\alpha_{95}^2}{\Sigma\theta_i^2} = \frac{R\alpha_{95}^2}{2(\phi/2)^2}$$

where ϕ is defined by:

$$R = 2\cos(\phi/2).$$

Assuming, for example, that $\hat{\delta} = 20°$, the value of R obtained with the aid of equation (13) is 1.93908 and the corresponding value of $\phi/2$ is 14.1779°. Replacing these values in the last equation yields the value of 62.76° for α_{95}. This is almost identical to the value of α_{95} yielded by equation (37) — 62.60° — but it is nearly 3.5° smaller than the value yielded by Fisher's solution — 66.23°. It appears therefore that equation (37) is perhaps more than a valid alternative to Fisher's solution.

THE MULTISAMPLE SET OF DATA

A logical extension to the above analysis is the description of a method to determine the most reliable indicator of a vector population mean direction and the corresponding limit of confidence, given N vectors drawn from that population in B independent samples.

This situation is encountered in paleomagnetic research when a small number (usually 4 or 5) of oriented specimens are collected from each of a number (B) of widely separated geographic localities. The question arises then as to whether or not the geographic origin of a specimen imposes any systematic bias on the orientation of its magnetization and, if so, whether or not the resultant R of the N magnetization direction unit vectors may still be considered as the best indicator of the population mean direction of magnetization. If the answer to the second question is negative, then how should the limit of confidence about the most reliable indicator be determined?

It is noted first that the resultant R_j of the N_j unit vectors in the j^{th} sample may be considered individually as an indicator of the population mean direction and that:

$$\hat{\delta}_j^2 = \frac{2(N_j-R_j)}{(N_j-1)}$$

is a valid estimator of the population angular variance. But since these statistics for any one of the B samples only use part of the available information, they must be discarded as the most appropriate to define the characteristics of the population under study.

Noting however that

$$\hat{\delta}_j^2 = \hat{\delta}_w^2 + \frac{R_j}{N_j}\hat{\delta}_{Rj}^2$$

where $\hat{\delta}_w^2$ relates to the dispersion of the unit vectors in the j^{th} sample about their resultant and $\hat{\delta}_{Rj}^2$ refers to the deviation of the latter from the population mean direction, we may formulate further the null hypothesis that all B resultants are parallel or nearly so. If such were the case, the resultants would also be parallel to the population mean direction, in which case all values of $\hat{\delta}_{Rj}^2$ would be zero. We could then write:

$$\Sigma(N_j - 1)\hat{\delta}_w^2 = 2\Sigma(N_j-R_j)$$

or

$$\hat{\delta}^2_w = \frac{2(N-\Sigma R_j)}{(N-B)} \qquad (38)$$

and refer to $\hat{\delta}^2_w$ as the pooled estimator of the within-sample angular variance.

Defining the angular variance estimator for the population by:

$$N\hat{\delta}^2 = N\hat{\delta}^2_w + B\hat{\delta}^2_\xi \qquad (39)$$

where $\hat{\delta}^2_\xi$ is the angular variance estimator for the population of sample mean directions (i.e. of the unit vectors respectively parallel to the resultants), it is obvious that $\hat{\delta}^2_w$ will only be a valid estimator of δ^2 when $\hat{\delta}^2_\xi$ is negligibly small. A second equation involving both $\hat{\delta}^2_w$ and $\hat{\delta}^2_\xi$ is therefore required to test the validity of the above null hypothesis. We may then write

$$R = \Sigma R_j \cos\beta_j$$

where β_j is the angle between the j^{th} resultant and R, the common resultant to both the ΣN_j unit vectors and the B sample resultants. Noting that in the present context we may write

$$\cos\beta_j = 1 - \beta_j^2/2,$$

the last equation becomes

$$R = \Sigma R_j - \Sigma R_j\, \beta_j^2/2$$

or

$$\frac{2(\Sigma R_j - R)}{B} = \frac{\Sigma R_j \beta_j^2}{B} = d_b^2$$

where d_b^2 is identified as the angular variance of the sample made up of the B sample resultants. This, however, is a biased estimator of the angular variance of the population of sample resultants, δ_b^2, since it relates to R instead of the population mean direction. The bias in d_b^2 may be seen to be directly proportional to δ_b^2 and inversely proportional to B so that, denoting the unbiased estimator of δ_b^2 by $\hat{\delta}_b^2$, we may write:

$$\hat{\delta}^2_b = d_b^2 + \hat{\delta}^2_b/B$$

or

$$\hat{\delta}^2_b = \frac{2(\Sigma R_j - R)}{(B-1)} \qquad (40)$$

This estimator clearly takes into account the dispersion of the unit vectors about their respective resultants (as referred to by the values of R_j) and also that of the sample resultants about R (as referred to by the difference ΣR_j-R), and thus we may write:

$$\hat{\delta}^2_b = \hat{\delta}^2_w + \frac{\Sigma R_j}{B}\,\hat{\delta}^2_\xi, \quad \hat{\delta}^2_\xi > 0 \qquad (41)$$

In order to test the validity of the null hypothesis, we may then compare $\hat{\delta}^2_b$ with $\hat{\delta}^2_w$. If these are equal or nearly so, the implication is that $\hat{\delta}^2_\xi$ is negligible and that $\hat{\delta}^2_w$ is a valid estimator of δ^2; if they differ significantly, then $\hat{\delta}^2_\xi$ is no longer to be ignored and may be determined by combining equations (38) and (41). Introducing this value of $\hat{\delta}^2_\xi$ into equation (39) yields the appropriate estimator for the unit vector angular variance.

Noting again that in the present context

$$\cos x = 1 - x^2/2$$

we may derive from equations (38) and (40) respectively:

$$\cos\hat{\delta}_w = \frac{(\Sigma R_j - B)}{(N-B)} \qquad (42)$$

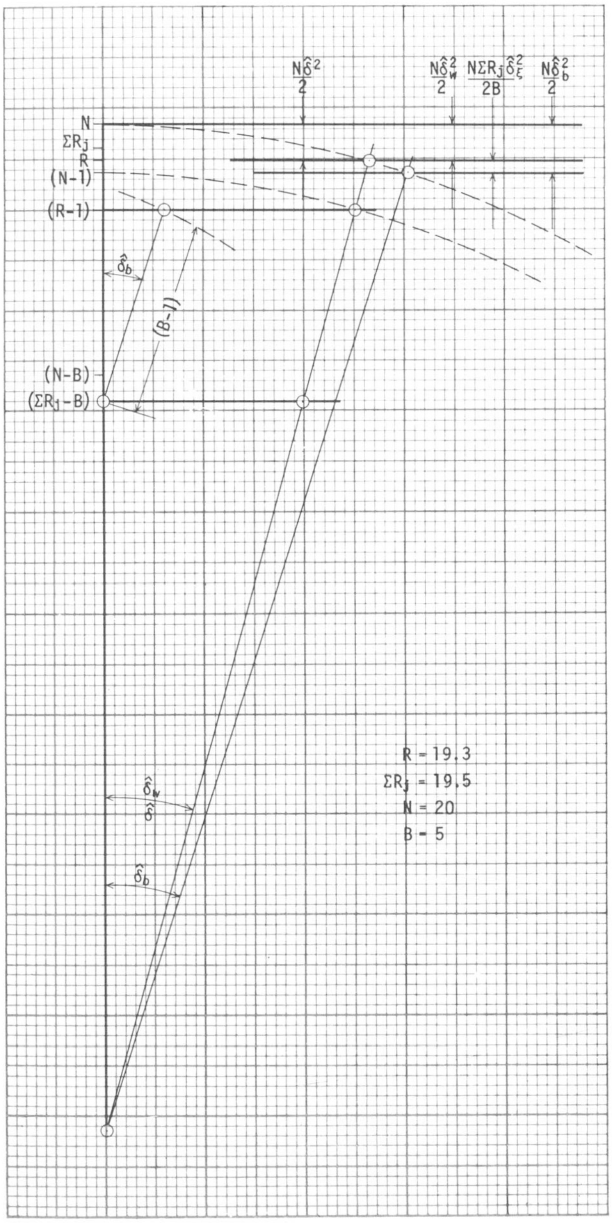

Figure 3. Graphical representation of data set characterized by N=20, B=5, R=19.3, ΣR_j=19.5.

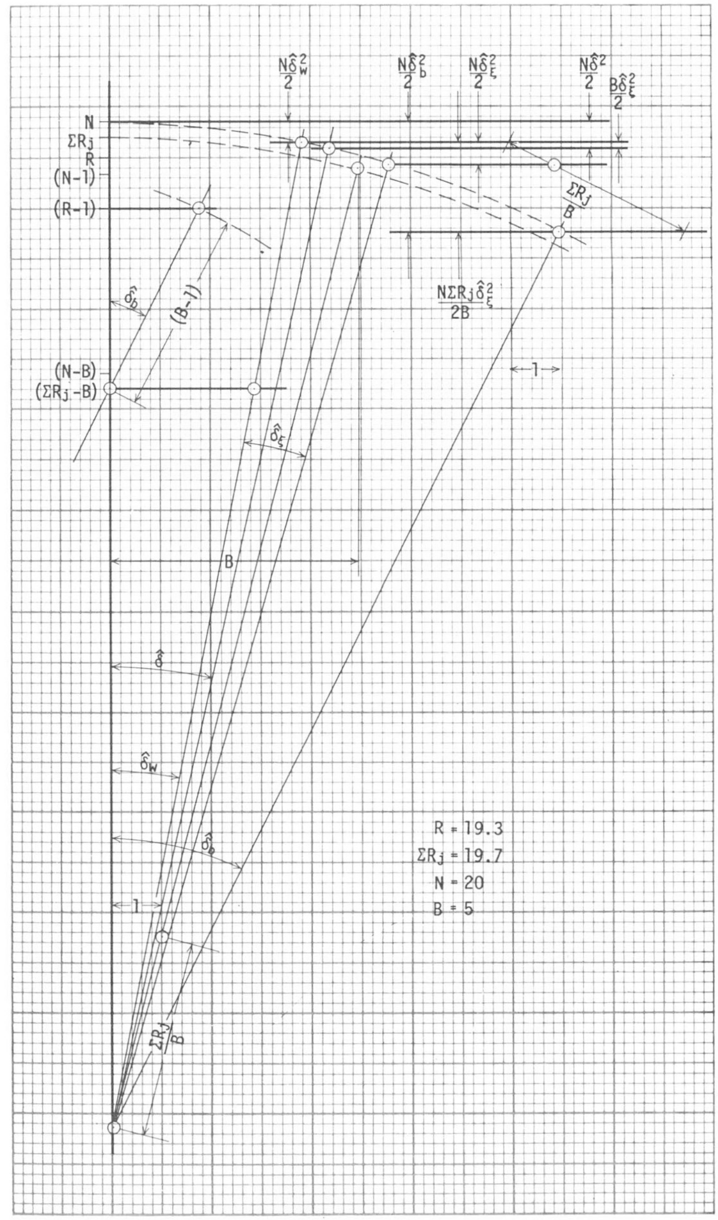

Figure 4. Graphical representation of data set characterized by N=20, B=5, R=19.3, ΣR_j=19.7.

and

$$\cos\hat{\delta}_b = \frac{(R-1) - (\Sigma R_j - B)}{(B-1)} \tag{43}$$

The last two equations may be used to draw the geometric representations of $\hat{\delta}$, $\hat{\delta}_w$ and $\hat{\delta}_b$ and to illustrate the relationships between $\hat{\delta}^2$, $\hat{\delta}_w^2$, $\hat{\delta}_b^2$ and $\hat{\delta}_\xi^2$, for any given data set.

Figure 3 represents a hypothetical set of 20 unit vectors evenly distributed between 5 independent samples for which ΣR_j and R are assumed to be equal to 19.5 and 19.3 respectively. No appreciable difference may be represented graphically between $\hat{\delta}_w$ and $\hat{\delta}$, since $\hat{\delta}_w^2$ is nearly equal to $\hat{\delta}_b^2$; $\hat{\delta}_\xi^2$ may be considered as negligible. Since the dispersion of the sample mean directions is very small the best indicator of the population mean direction is either the resultant of the N vectors, R, or that of the B sample mean directions. The limit of confidence about the sample mean indicator at the 0.95 probability level is then given by equations (36) or (37).

The hypothetical set of data represented in Figure 4 only differs from that represented in Figure 3 by the assumption that ΣR_j is equal to 19.7 instead of 19.5. A comparison of the two diagrams indicates readily, however, that the compositions of the two sets of data are fundamentally different.

It is observed in Figure 4 that $\hat{\delta}^2_\xi$ is approximately equal to $\hat{\delta}^2_w$ and therefore cannot be considered as negligible, as in Figure 3. It is also noted that $\hat{\delta}_w$ is smaller than $\hat{\delta}$, which is in turn smaller than its counterpart shown in Figure 3, even though the value of R is the same in both cases. This is to be expected, however, because the "geographic bias" in the second case tends to cluster the vectors together within their respective samples.

It would be misleading to use R as the indicator of the population mean direction in the second case because its orientation is partly biased by those samples characterized by the lower dispersions and/or, should the number of the unit vectors be unequal in all B samples, by the larger samples.

On the other hand, the resultant "r" of the B sample mean directions (i.e. of the B unit vectors respectively parallel to the B sample resultants) is not biased either by the relative dispersions of the individual samples or by the actual sizes of the samples. Because it also "averages out" the geographic bias in each sample, it appears to be the most appropriate indicator of the unit vector population mean direction.

The adequate limit of confidence about r is given by:

$$\alpha_{95} = \frac{\sqrt{3r}\,\hat{\delta}_\xi}{(r-1)} \simeq \sqrt{F^*}\,\frac{\hat{\delta}_\xi}{\sqrt{r}} \qquad (44)$$

where $\hat{\delta}_\xi$ may be computed by replacing the calculated values of $\hat{\delta}^2_w$ and $\hat{\delta}^2_b$ in equation (41).

An approximate value of α_{95} may be obtained by replacing $\hat{\delta}_\xi$ in the last equation by $\hat{\delta}_m$, which is given by

$$\cos\hat{\delta}_m = \frac{(r-1)}{(B-1)}$$

This estimator of the population of sample mean directions standard deviation is less accurate, however, because it ignores the effect of the within-site dispersion.

In this specific example, the error committed in using R instead of r as the population mean direction indicator may not be important as such, but the misleading effect of using equations (36) or (37) instead of equation (44) to calculate the confidence limit translates into the difference between the exaggeratedly low value of 5.3° and the correct value of 11.3° for α_{95}.

Even a casual comparison of these two graphical representations indicates that there is no "geographic bias" in the first case while there is an obvious one in the second. It is not so evident "a priori" to decide which of R or r should be used as the sample population mean direction indicator and to choose the appropriate formula for the calculation of the limit of confidence for an intermediate value of ΣR_j between 19.5 and 19.7. Fortunately, Watson (1956) has given a mathematical solution to this problem by demonstrating the legitimacy of transposing the F-ratio test of significance (which is currently used in Gaussian statistics) into directional statistics.

The object of the test is to establish for a chosen probability level (usually 0.05) the maximum value of the ratio $\hat{\delta}^2_b/\hat{\delta}^2_w$ for which $\hat{\delta}^2_\xi$ may be considered negligible. This maximum value corresponds to that of the F-distribution for 2(B-1), 2(N-B) degrees of freedom, at the chosen probability level.

The two hypothetical cases examined above may be used to illustrate the application of the F-ratio test. As there are 20 unit vectors and 5 independent samples in both cases, the common value of $F_{8,30,.05}$ given in the tables is 2.27. On the other hand, the ratio $\hat{\delta}^2_b/\hat{\delta}^2_w$ is 1.5 in the first case and 5.0 in the second. The test confirms, therefore, the indications of the two graphical representations, i.e. that $\hat{\delta}^2_\xi$ is not significant in the first case and significant in the second. The quantitative implication is that the probability is less than 0.05 (in fact less than 0.005) that $\hat{\delta}^2_\xi$ is not significant in the second case whereas the same probability is greater than 0.05 (in fact greater than 0.1) in the first case. In practical terms, this suggests that there is probably no "geographic bias" in the first case and that there is one in the second.

A corollary to the above considerations relates to the design of further sampling in both cases, should it be felt necessary to increase the precision of knowledge on the population mean direction. Obviously, it would be quite adequate to simply obtain additional oriented specimens from any one of the already sampled localities in the first case. On the other hand, this approach would be of little use in the second case, where adequate sampling at additional localities would be required to reduce the factor $\sqrt{r}/(r-1)$ in equation (44).

CONCLUSIONS

The title of this paper implies that there is an alternative to Fisher's solution for the calculation of the limit of confidence relating to the resultant of a unit vector sample. The results listed in Tables 4 and 5 show that equations (36) and (37) may be regarded as such alternatives, at least for all practical purposes in paleomagnetic research.

In addition, the derivation from first principles of the universally accepted angular deviation density function for unit vector populations indicates that the latter simulates most adequately the Gaussian probability function in directional statistics, provided the angular standard deviation of the population under study does not exceed certain limits which appear to be much lower than $\pi/4$. Beyond that limit, the angular density function loses its relevance gradually and so does the meaning of a limit of confidence, independently of the solution used to calculate it. Similar restrictions will clearly apply to the validity of transposing directly certain tests of significance from the Gaussian statistical system into the directional statistical system.

Mathematicians have discouraged in the past the use of the angular deviation notation to define the angular dispersion of a unit vector population. Their preference for the familiar clustering index "k" appears to stem from the fact that the angular deviation notation would allegedly lead away from the vectorial treatment.

Nevertheless, there are certain advantages for the paleomagnetist and the casual reader of the paleomagnetic literature to picture such abstract concepts as standard deviation and variance in terms of degrees instead of its reciprocal to the half power. For example, one could easily be misled to believe that a sample yielding a k of 500 is considerably more coherent, and therefore credible, than another sample yielding a k of only 400. In fact the difference translates by less than 0.5° in terms of standard deviation index. On the other hand, there is a difference of 2.8° between the standard deviation indexes of two samples yielding k's of 15 and 20 respectively.

The angular deviation notation also lends itself to the graphical representation of a sample's statistics and as such leads to the visual representation of d, $\hat{\delta}$, and $\hat{\delta}_R$ as well as to the physical meaning of the various components of angular variance in multisample situations.

Finally, the angular deviation notation is perhaps more apt to suggest to the nonstatistician the similarities between the Gaussian and the directional statistical systems and thereby should facilitate the identification of possibilities to transpose into the latter certain tests of significance which are currently used in the Gaussian system.

REFERENCES

Arnold, K.J.
1941: On spherical probability distribution; Ph.D. Thesis, Massachusetts Institute of Technology.

Fisher, R.A.
1953: Dispersion on a sphere; Proceedings of the Royal Society of London, Series A, v. 217, 7 p.

Langevin, D.
1905: Magnétisme et théorie des électronics; Annales de Chimie et de Physique, v. 5, p. 70-127.

Watson, G.S.
1956: Analysis of dispersion on a sphere; Monthly Notice of the Royal Astronomical Society, Geophysical Supplement, v. 7, no. 4, p. 153-159.

Watson, G.S. and Irving, E.
1957: Statistical methods in rock magnetism; Monthly Notice of the Royal Astronomical Society, Geophysical Supplement, v. 7, no. 6, p. 291-300.