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GROUNDWATER FLOW-DIAGRAMS IN SECTIONS WITH EXAGGERATED VERTICAL SCALE

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R. O. van Everdingen

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By

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GROUNDWATER FLOW-DIAGRAMS IN SECTIONS WITH EXAGGERATED VERTICAL SCALE

Introduction

Vertical cross-sections showing qualitative flow-diagrams for groundwater movement may be used to illustrate results of groundwater investigations. If large areas are represented in hydrogeological sections it appears to be convenient to exaggerate the vertical scale of such sections, a practice that enables the presentation of more geological detail. It should be stressed, however, that such exaggeration of the vertical scale distorts the flow-pattern in the section, a fact that is not always realized. Because of the distortion, the flowlines (or flow-direction lines) and the equipotential lines in an isotropic medium in the section are no longer orthogonally intersecting.

Assuming a section in which the equipotentials have been constructed (drawn) from actual field-data, the amount of deviation from the orthogonal pattern can be determined for various exaggeration factors.

Notations used are:

X = exaggeration factor, ratio of vertical and horizontal scale.

α = acute angle between an equipotential line and the horizontal, in the true-scale section.

α_1 = the corresponding angle in the exaggerated section.

β = acute angle between the corresponding flowline and the horizontal, in the true-scale section.

β_1 = the same angle in the exaggerated section.

ϕ = angle between equipotential line and corresponding flowline, in the true-scale section.

ϕ_1 = the corresponding angle in the exaggerated section ($\leqslant 90°$).

Construction of Flowlines

For the general case we assume α_1 and X to be known. Figure 1 shows the true-scale and exaggerated sections for such a case. Comparison of the two sections shows:

$$\tan \alpha = \frac{1}{X} \cdot \tan \alpha_1 \qquad (1)$$

$$\tan \beta = X.\ \cot \alpha_1 \qquad (2)$$

$$\tan \beta_1 = X^2.\ \cot \alpha_1 \qquad (3)$$

$$\tan \gamma_1 = \frac{1}{X^2}\ .\ \tan \alpha_1 \qquad (4)$$

$$\tan \delta_1 = \cot \alpha_1 \qquad (5)$$

Now $\phi_1 = \gamma_1 + \delta_1$ which gives:

$$\tan \phi_1 = \tan\ (\gamma_1 + \delta_1) = \frac{\tan \gamma_1 + \tan \delta_1}{1 - \tan \gamma_1 .\ \tan \delta_1} \qquad (6)$$

Combination of (6) with (4) and (5) gives:

$$\tan \phi_1 = \frac{\frac{1}{X^2}\ \tan \alpha_1 + \cot \alpha_1}{1 - \frac{1}{X^2}\ \tan \alpha_1 .\ \cot \alpha_1} = \frac{\tan \alpha_1 + X^2.\ \cot \alpha_1}{X^2 - 1} \qquad (7)$$

It is clear from (7) that ϕ_1 will be equal to 90° only when α_1 = 0° or α_1 = 90°, and for X = 1. In all other cases, $\phi_1 < 90°$ will be valid.

The derivative of $\tan \phi_1$ with respect to α_1, for any particular value of X, is

$$\frac{d \tan \phi_1}{d \alpha_1} = \frac{\frac{1}{\cos^2 \alpha_1} - \frac{X^2}{\sin^2 \alpha_1}}{X^2 - 1} \qquad (8)$$

We find the minimum angle ϕ_1 for any X when (8) is equal to zero. This occurs when:

$$\frac{1}{\cos^2 \alpha_1} = \frac{X^2}{\sin^2 \alpha_1} \qquad (9)$$

Then: $$\tan^2 \alpha_1 = X^2 \quad \text{or} \quad \tan \alpha_1 = X \qquad (10)$$

Thus the angle between flowlines and equipotential lines in a section with exaggerated vertical scale is minimal for a given X, when α_1 is such that $\tan \alpha_1 = X$. Thus, for X = 10 the angle ϕ_1 is smallest when $\tan \alpha_1 = 10$ or $\alpha_1 = 84°17'$. The angle ϕ_1 in that case is 11°25' and the angle $\beta_1 = 84°17'$. This means that β_1 decreases from 84°17' to 0° for an increase of α_1 from 84°17' to 90°, or a change in β_1 of 84°17' for a change in α_1 of only 5°43'. Large inaccuracies are therefore unavoidable in this case. For this reason vertical exaggerations of more than 10 times should be avoided.

The construction of flow-diagrams in sections with exaggerated vertical scale can be facilitated by the use of a set of curves from which the values of the various angles can easily be determined for a given exaggeration factor. Three nomographs were constructed for this purpose. Standard exaggeration factors used for the curves are 1.5; 2; 3; 4; 5; 7; 10; and 15.

The first nomograph (Figure 2) shows the curves for the determination of ϕ_1 from α_1 and X (equation 7). The second nomograph (Figure 3) shows the curves for the determination of β_1 from α_1 and X (equation 3). The third nomograph (Figure 4) enables determination of α and β from α_1 and X (equations 1 and 2).

It is not necessary to use both the nomographs of Figures 2 and 3. Once ϕ_1 is determined, β_1 can be found from $\beta_1 = 180° - (\phi_1 + \alpha_1)$. If β_1 is determined first, ϕ_1 is found from the same relationship. The symmetry of the curves in Figure 3 shows that the flowlines and equipotential lines remain interchangeable in sections with exaggerated vertical scale.

When these nomographs are used for the construction of flowlines from equipotential lines in a section with exaggerated vertical scale, it is subsequently possible to account for the distortion and to determine the actual direction of flow. The following two examples illustrate the importance of correcting for the vertical exaggeration.

The first example consists of part of a flow-diagram constructed by Meyboom (1962, Fig. 1, section C-D)[1]. In this section (Fig. 5A) groundwater flow takes place in a stratified medium, consisting of a poorly permeable upper layer and a permeable lower layer. Meyboom constructed equipotentials on the basis of actual field-data. Flowlines - corresponding to the flow-direction arrows in Meyboom's original section - are constructed at right angles to the equipotential lines. Vertical exaggeration of the section is approximately 15 times. Figure 5B shows the flow-pattern constructed with the use of the nomograph in Figure 3. It is apparent, that Figure 5A suggests more lateral flow in the upper part of the section than actually exists. Figure 5B shows the lateral flow to be largely concentrated in

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[1] Dates in parentheses are those of publications listed in the References.

the more permeable zone of the Cretaceous bedrock.

The second example consists of a flow-diagram by Toth (1962, Fig. 6). Groundwater flow takes place in a semi-homogeneous medium, containing one highly permeable lens. Here both equipotentials and flowlines were constructed by Toth, based on mutual orthogonality (see Fig. 6A). The exaggeration of the vertical scale is also approximately 15 times. Figure 6B shows the flow-pattern constructed with the use of the nomograph in Figure 3. The corrected pattern shows that only a minor portion of the flow in the section is lateral, whereas the bulk of the lateral flow apparently takes place below the lowest level shown.

In both examples only minor changes occur in the distribution of recharge and discharge areas, although discharge in the corrected section becomes more concentrated in the central part of the discharge area.

Construction of Flow-net Squares

The next step is to investigate the possibility of representing groundwater movement by means of a flow-net in the exaggerated section. In a flow-net on true-scale constructed for a homogeneous isotropic medium the flowlines and equipotential lines form so-called curvilinear squares, i.e. squares with curved sides. The four sides of such a square will only be equal in length (and straight) when the equipotential lines are straight and parallel. In the general case all four sides of a square may be different in length, on the condition that the mean of the two sides on the flowlines is equal to the mean of the two sides on the equipotential lines.

Figure 7 shows both the true-scale and the vertically exaggerated sections for the case where two equipotential lines in the exaggerated section are parallel and straight.

In the true-scale section (Fig. 7A) the sides e of the square along the equipotentials are equal in length to the sides f along the flowlines. The problem consists of expressing e_1 (= C_1B_1) in the knowns α_1, β_1, ϕ_1, and f_1 (= A_1B_1). In the triangle $C_1B_1F_1$ in the exaggerated section:

$$(C_1B_1)^2 = (C_1F_1)^2 + (B_1F_1)^2 \qquad (11)$$

Comparison of the true-scale and exaggerated sections shows:

$$(C_1F_1) = (CF) \text{ (horizontal, not distorted) and } (B_1F_1) = X.\ (BF) \qquad (12)$$

Combination of (11) and (12) gives:

$$(C_1B_1)^2 = (CF)^2 + X^2.\ (BF)^2 \qquad (13)$$

In triangle CBF in the true-scale section:

$$(CF) = (CB).\cos\alpha \text{ and } (BF) = (CB).\sin\alpha \qquad (14)$$

Substitution of (14) in (13) gives:

$$(C_1B_1) = (CB).\cos\alpha + (CB).X.\sin\alpha \qquad (15)$$

Introducing $(C_1B_1) = e_1$; $(CB) = e$; $(A_1B_1) = f_1$; $(AB) = f$ we find for (15):

$$e_1^2 = e^2.\cos^2\alpha + e^2.X^2.\sin^2\alpha \qquad (16)$$

In the same way we find for $A_1B_1 = f_1$:

$$f_1^2 = f^2.\sin^2\alpha + f^2.X^2.\cos^2\alpha \qquad (17)$$

Combination of (16) and (17) and substitution of e=f gives:

$$e_1 = f_1\sqrt{\frac{\cos^2\alpha + X^2.\sin^2\alpha}{\sin^2\alpha + X^2.\cos^2\alpha}} \qquad (18)$$

When f_1 is known and when α is derived from α_1 through the nomograph in Figure 4, the length of e_1 can be calculated, using equation 18.

In order to facilitate the determination of e_1, a set of curves (Fig. 8) was constructed for the relation between the ratio e_1/f_1 and the angle α_1. These curves are a combination of equation 18 and the curves of Figure 4.

If one of the standard exaggerations mentioned earlier is used, the determination of the angles and ratios is made easier by the use of the nomographs of Figures 9 - 16. Each of these figures applies to one value of X only and gives the curves for ϕ_1 versus α_1; β_1 versus α_1; α and β versus α_1; and e_1/f_1 versus α_1. It should be noted that the scale for e_1/f_1 is not the same on all nomographs. This was done in order to keep the curve for e_1/f_1 inside the same limits as the other curves.

If an exaggeration factor different from the given standards is used, the corresponding curves can easily be determined through interpolation between the curves for the two nearest standard values of X in Figures 2, 3, and 4.

Construction of Flow-nets

For a section with a given vertical exaggeration X, in which α_1 is known for a set of straight, parallel equipotential lines, the construction of equivalent squares is carried out as follows.

From the angle α_1 the angle ϕ_1 or β_1 is determined, using Figure 2 or 3. The corresponding flowline can then be drawn. Consequently the value for f_1 can be determined. Next the value for e_1 can be found from the known values of α_1 and f_1, using Figure 8. Thus the equivalent squares can be constructed in the exaggerated section.

When the equipotential lines are not parallel, but still straight, one flowline is constructed in full, using Figure 2 or Figure 3. Taking the value of f_1 on the flowline between the first two equipotentials, and the average of the angles α_1 for these two equipotentials, e_1 average can be determined. This average value for e_1 is then plotted on an auxiliary equipotential midway between the two original ones, in such a way that its length is divided into two equal parts by the original flowline. This procedure is repeated for each pair of adjacent equipotentials. If next the end points of the e_1 lines are connected by smooth curves, two series of equivalent squares are produced. The flow-net can be extended in both directions away from the original flowline by repeating the whole process.

When the equipotential lines are curved instead of straight, the above procedure can still be used, in combination with the necessary trial-and-error approximation.

Conclusions

1. The vertical scale of cross-sections in which flow-diagrams or flow-nets are to be constructed should not be exaggerated if the exaggeration is not absolutely necessary. If a section becomes very long because of the adherence of its author to the true-scale principle, it is better to reproduce it on a fold-out page than to exaggerate the vertical scale.

2. If vertical exaggeration is necessary, it should be kept to a bare minimum and not exceed 10 times.

3. When the exaggeration of the vertical scale is not taken into account during the construction of flow-diagrams, the result will be an erroneous concentration of lateral flow in the upper part of the section.

4. Construction of quantitative flow-nets in sections with exaggerated vertical scale is more time-consuming and more inaccurate than in true-scale sections. In addition it may be pointed out that lateral flow into or out of the plane of the section under study may make the accurate construction of quantitative flow-nets almost senseless. In other words, before embarking on flow-net construction in a vertically exaggerated section, one must be sure that the section line is parallel to the horizontal component of the groundwater flow in the area under study.

5. If an aquifer in the section is inhomogeneous but still isotropic, the qualitative flow-diagram construction, as given above, is still valid. The quantitative flow-net construction, however, is no longer valid in that case, because in the true-scale section the 'squares' will change to 'rectangles'. The rectangles are elongated in the direction of the flowlines where the permeability (transmissibility) increases, and in the direction of the equipotential lines where the permeability decreases.

If in addition the aquifer is anisotropic, both constructions fail, because the angle ϕ between flowlines and equipotential lines in the true-scale section is no longer equal to 90°. Therefore sections should be constructed to true scale if quantitative data are available on the anisotropy of the aquifers.

References

Meyboom, P.
1962: Patterns of Groundwater Flow in the Prairie Profile; Proc., 3rd Canadian Hydrology Symp., Calgary, Alta., Nov. 1962, preprint, 21 p.

Toth, J.
1962: A Theory of Groundwater Movement in Small Drainage Basins in Southern Alberta, Canada; J. Geophys. Research, vol. 67, No. 11, pp. 4375-4387.

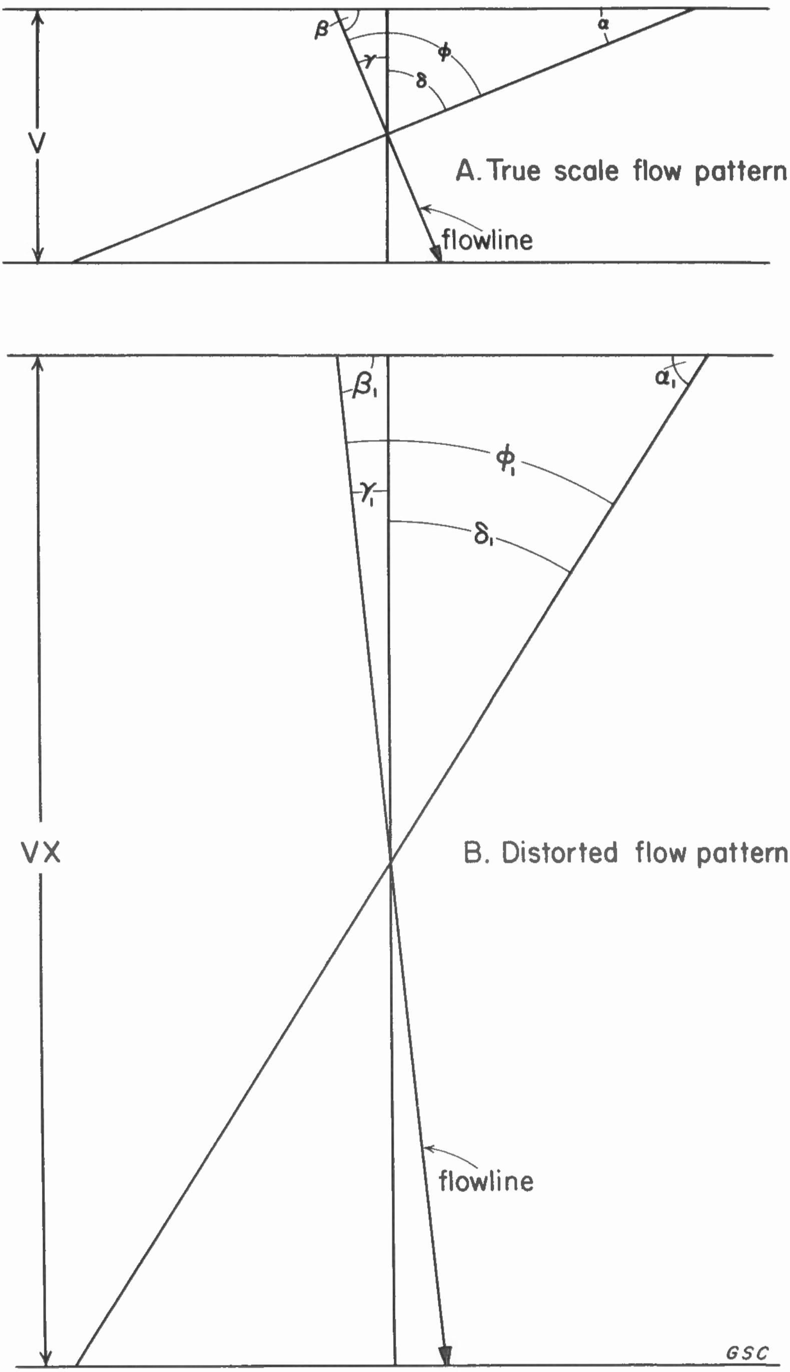

Figure 1. Relation of flowlines and equipotential lines in true-scale and distorted flow patterns

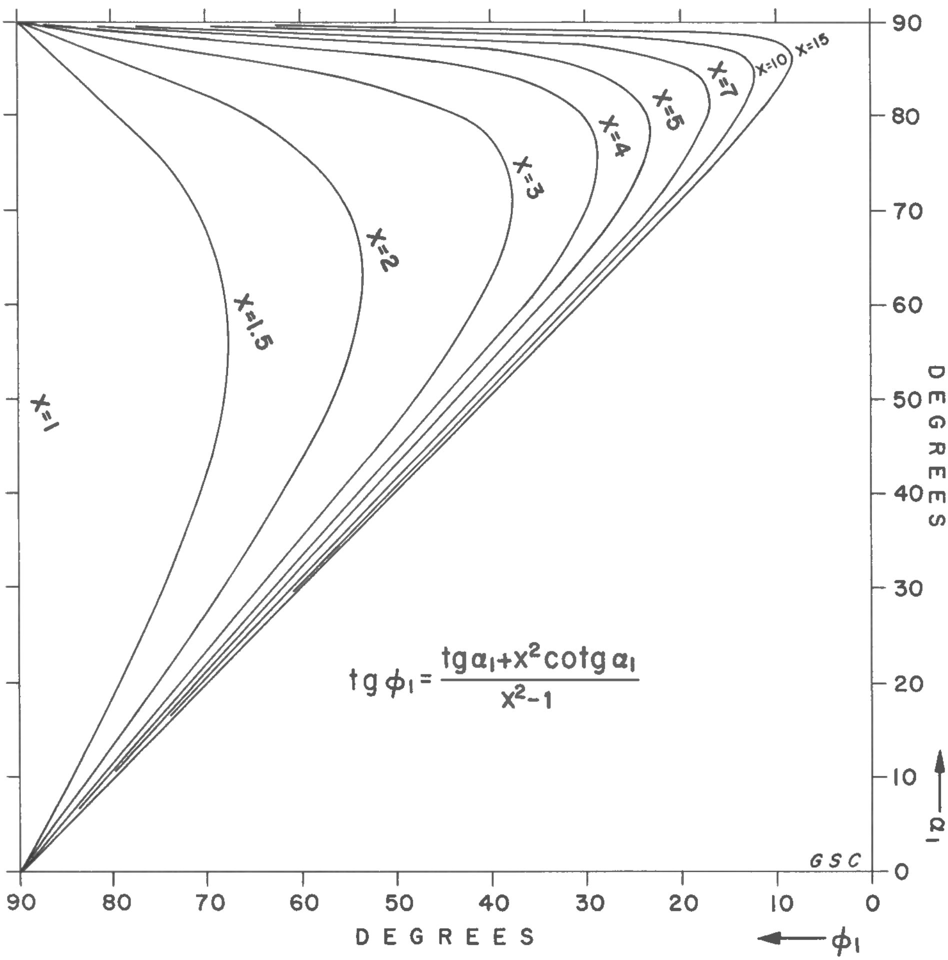

Figure 2. Nomograph for the determination of ϕ_1 from α_1 and X

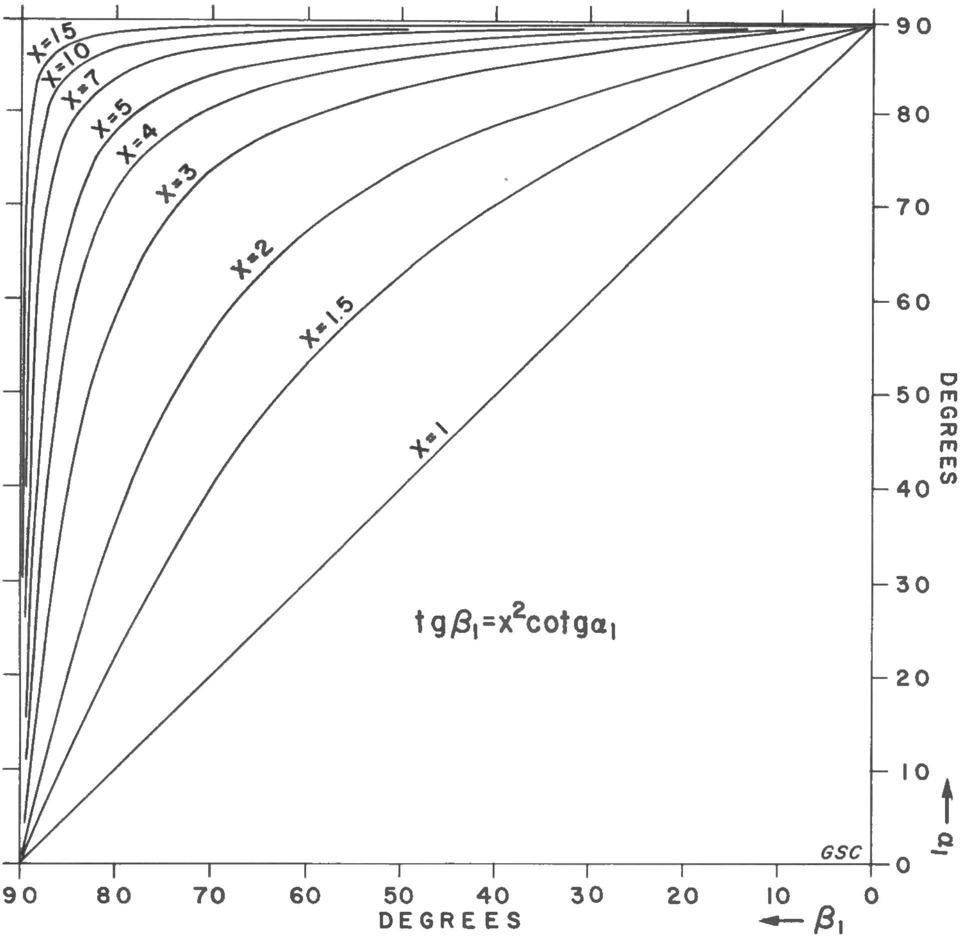

Figure 3. Nomograph for the determination of β_1 from α_1 and X

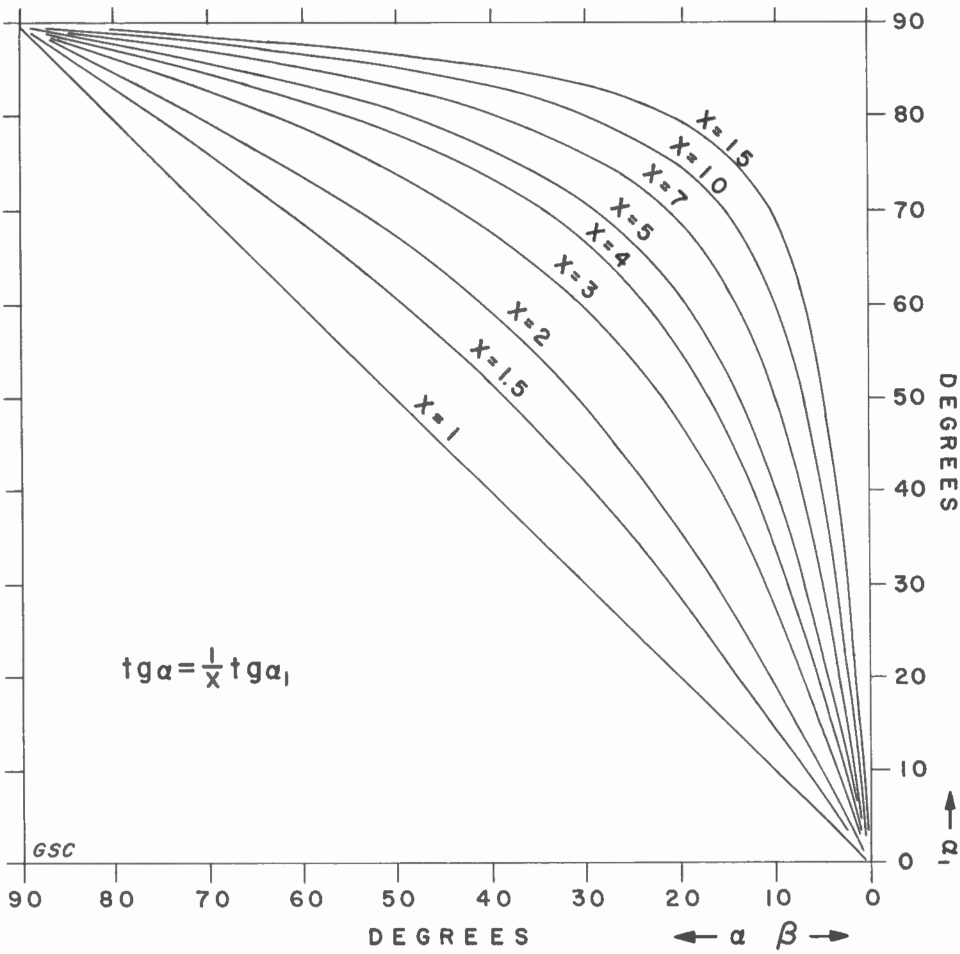

Figure 4. Nomograph for the determination of α and β_1 from α_1 and X

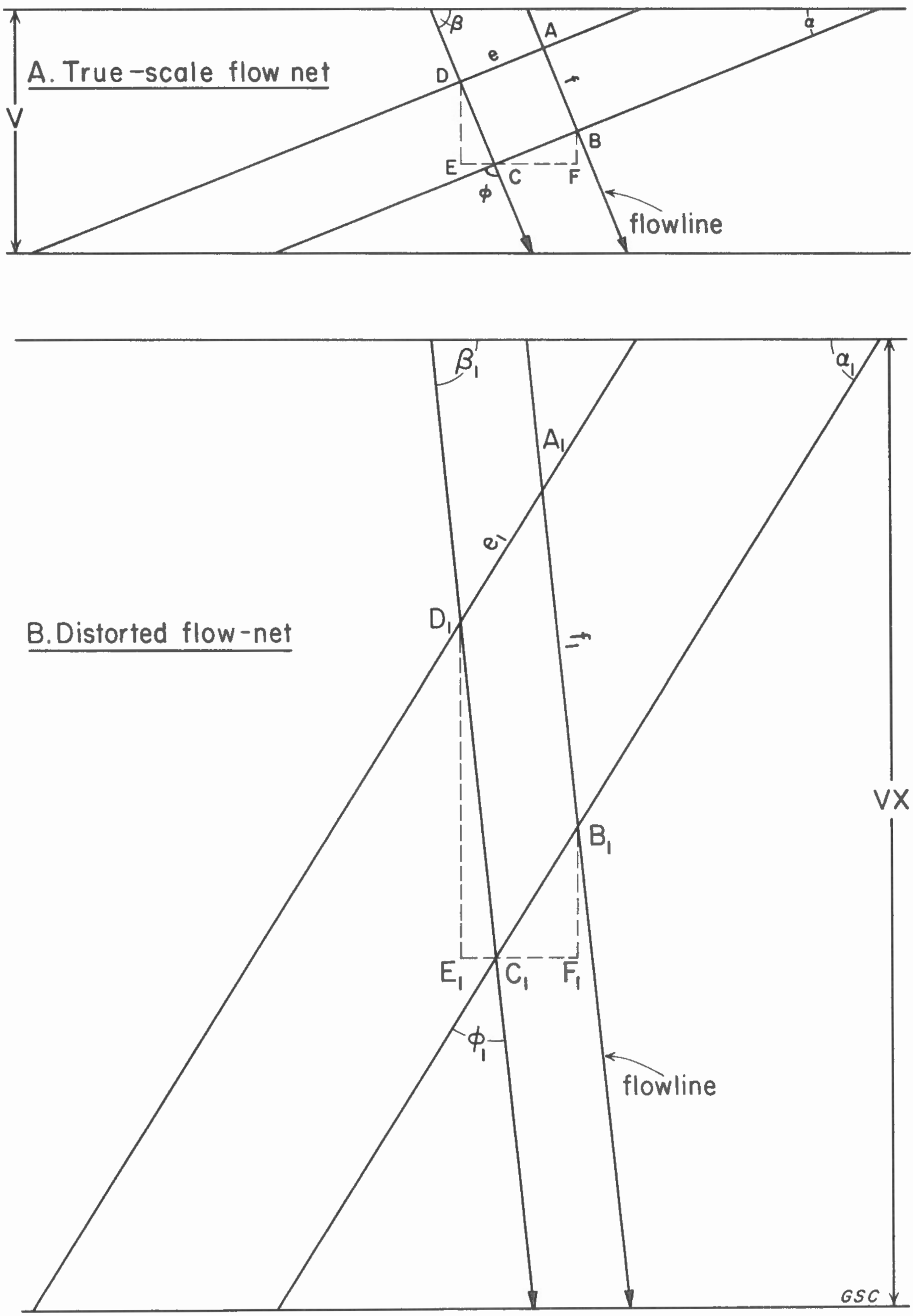

Figure 7. True-scale and distorted flow-net squares

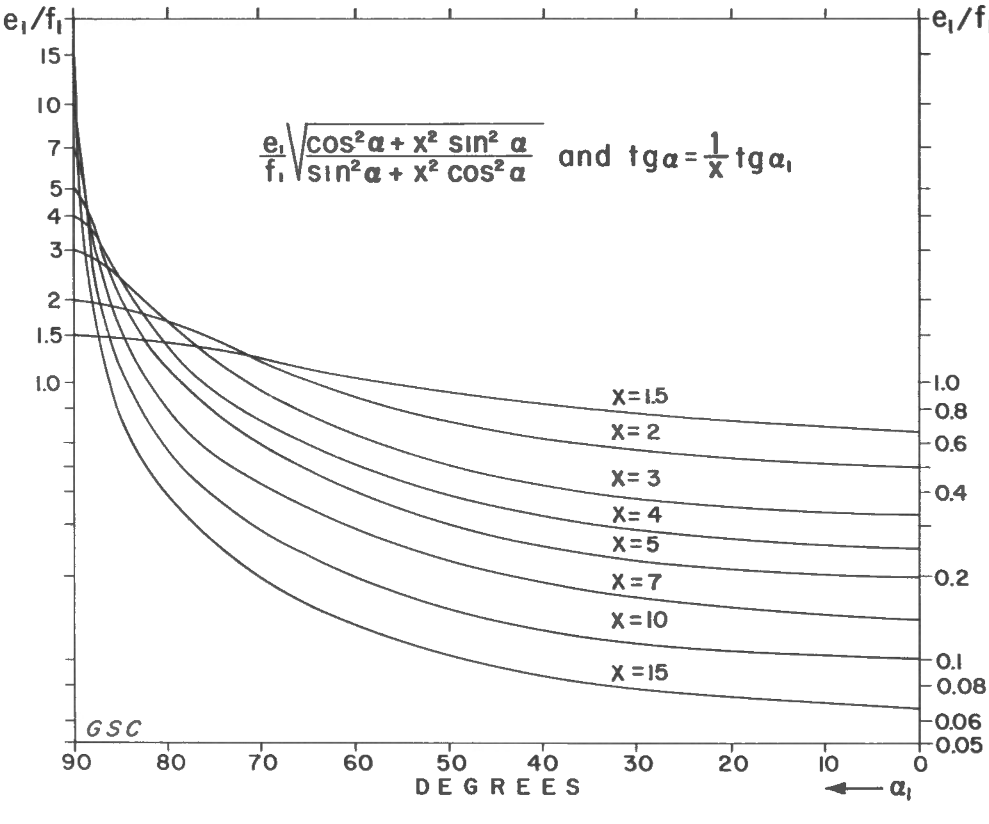

Figure 8. Nomograph for the determination of the ratio of e_1/f_1 from α_1 and X

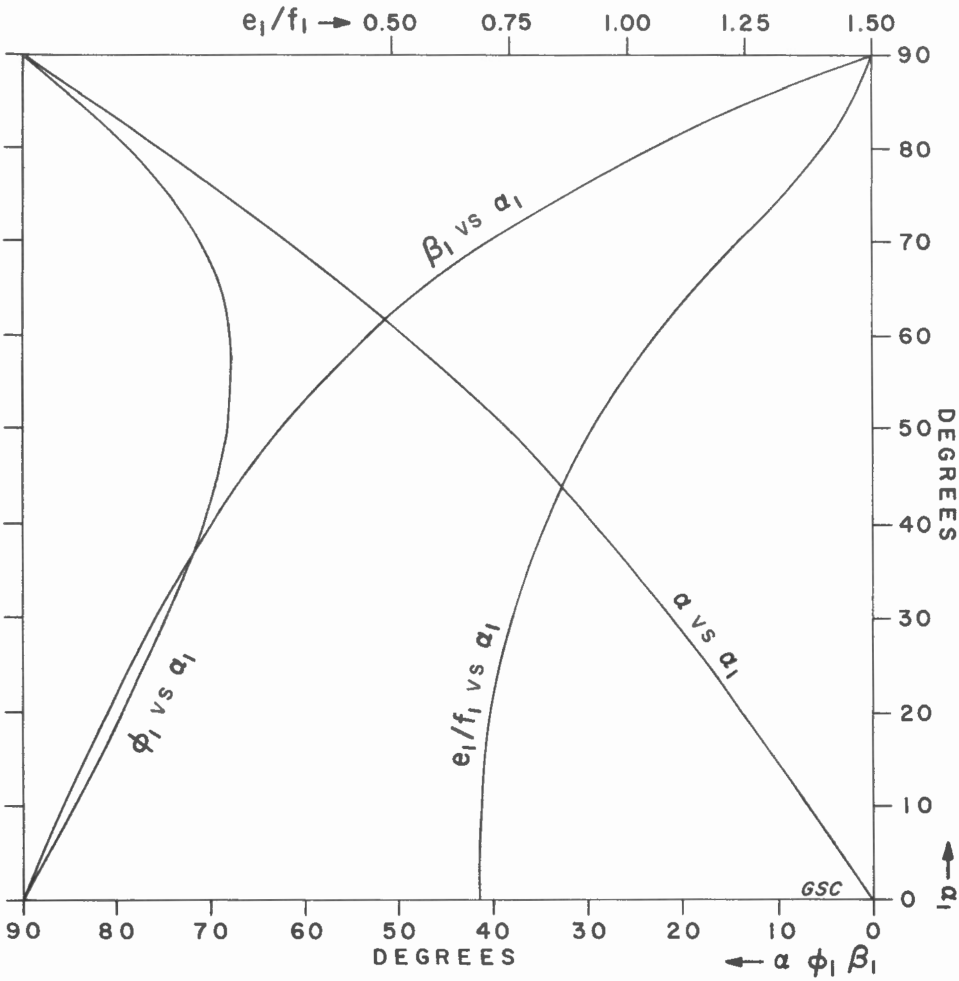

Figure 9. Nomograph for the determination of ϕ_1, β_1, α and e_1/f_1 from α_1 for $X = 1.5$

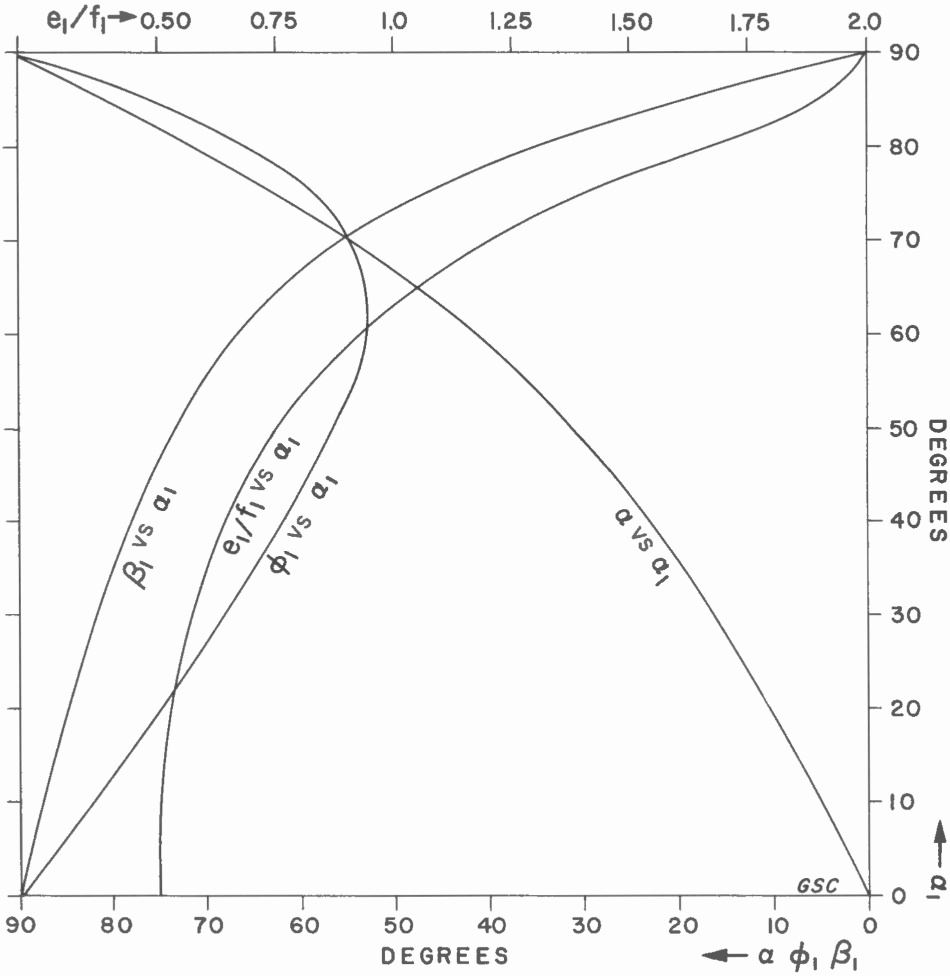

Figure 10. Nomograph for the determination of ϕ_1, β_1, α and e_1/f_1 from a_1 for $X=2$

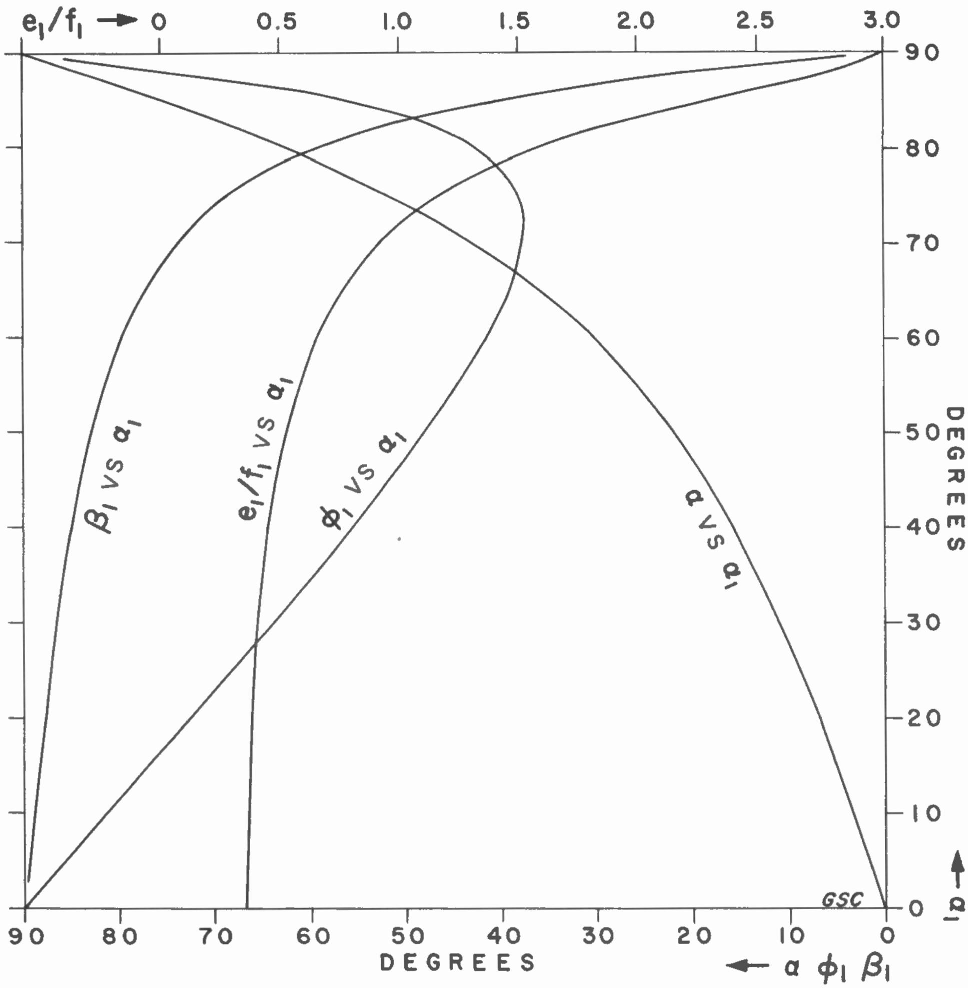

Figure 11. Nomograph for the determination of ϕ_1, β_1, α and e_1/f_1 from α_1 for $X=3$

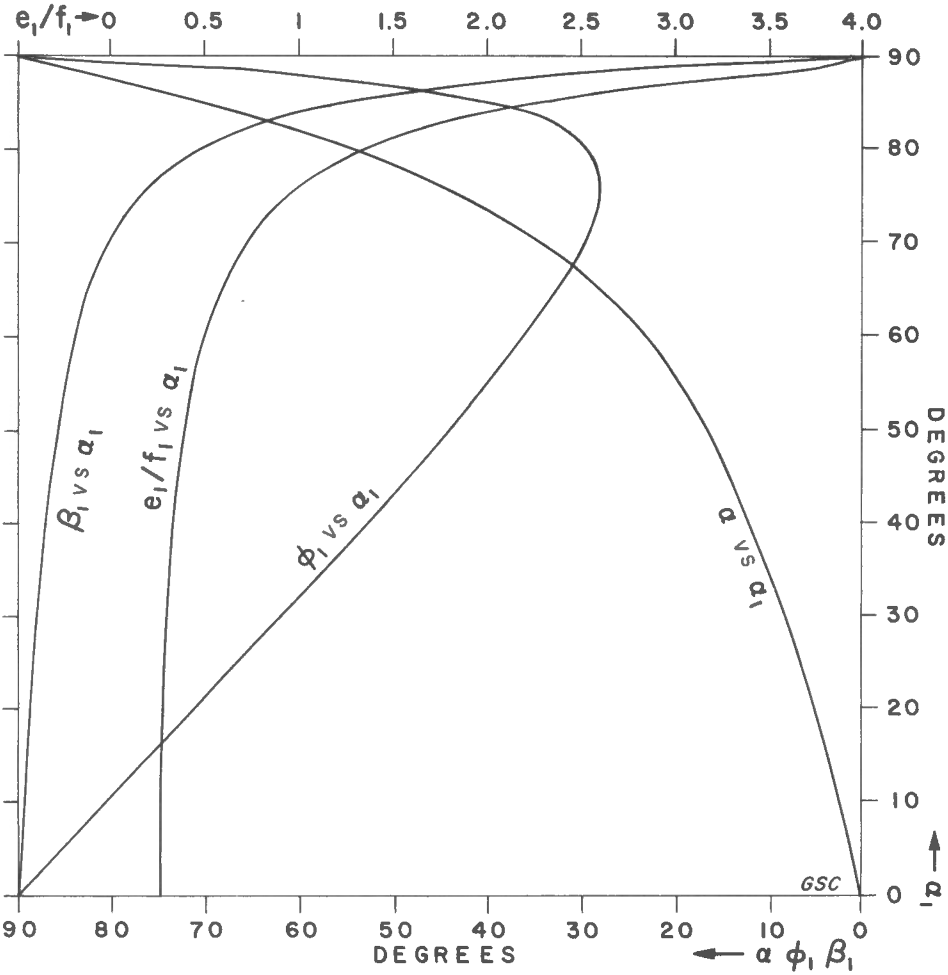

Figure 12. Nomograph for the determination of ϕ_1, β_1, α and e_1/f_1 from α_1 for $X = 4$

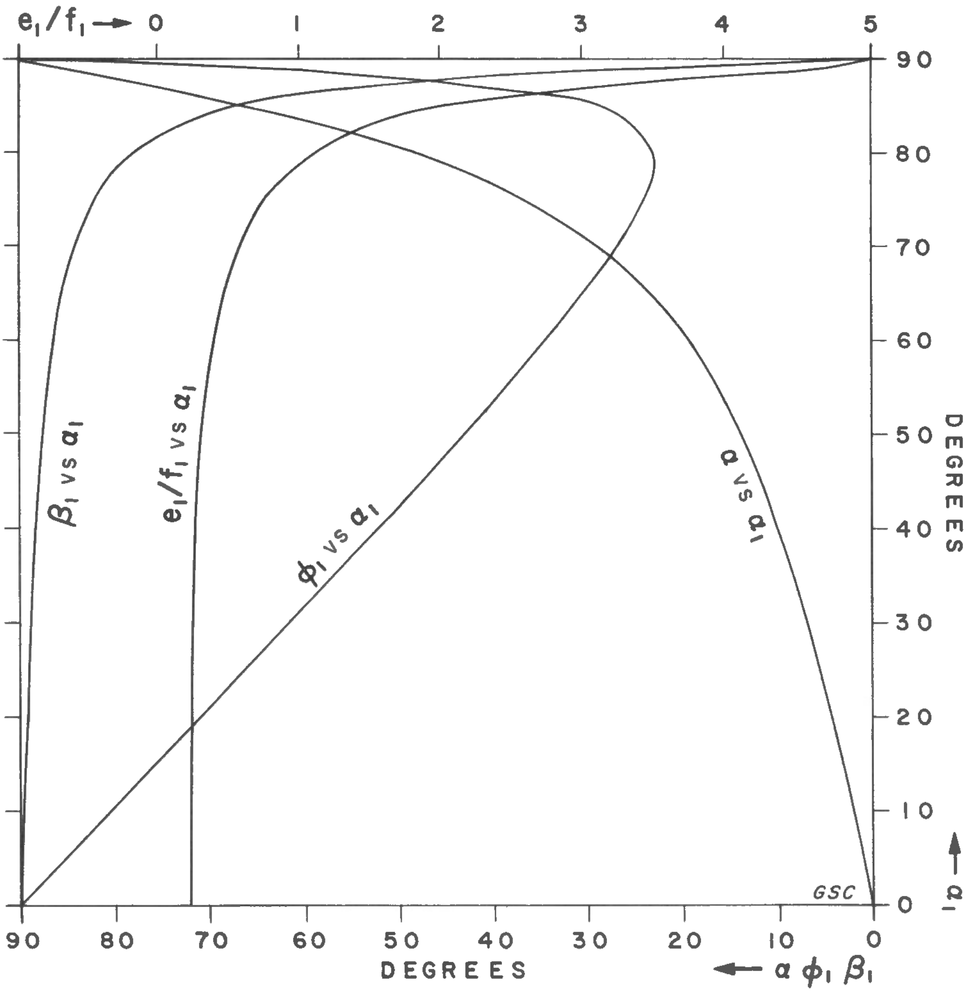

Figure 13. Nomograph for the determination of ϕ_1, β_1, α and e_1/f_1 from α_1 for X=5

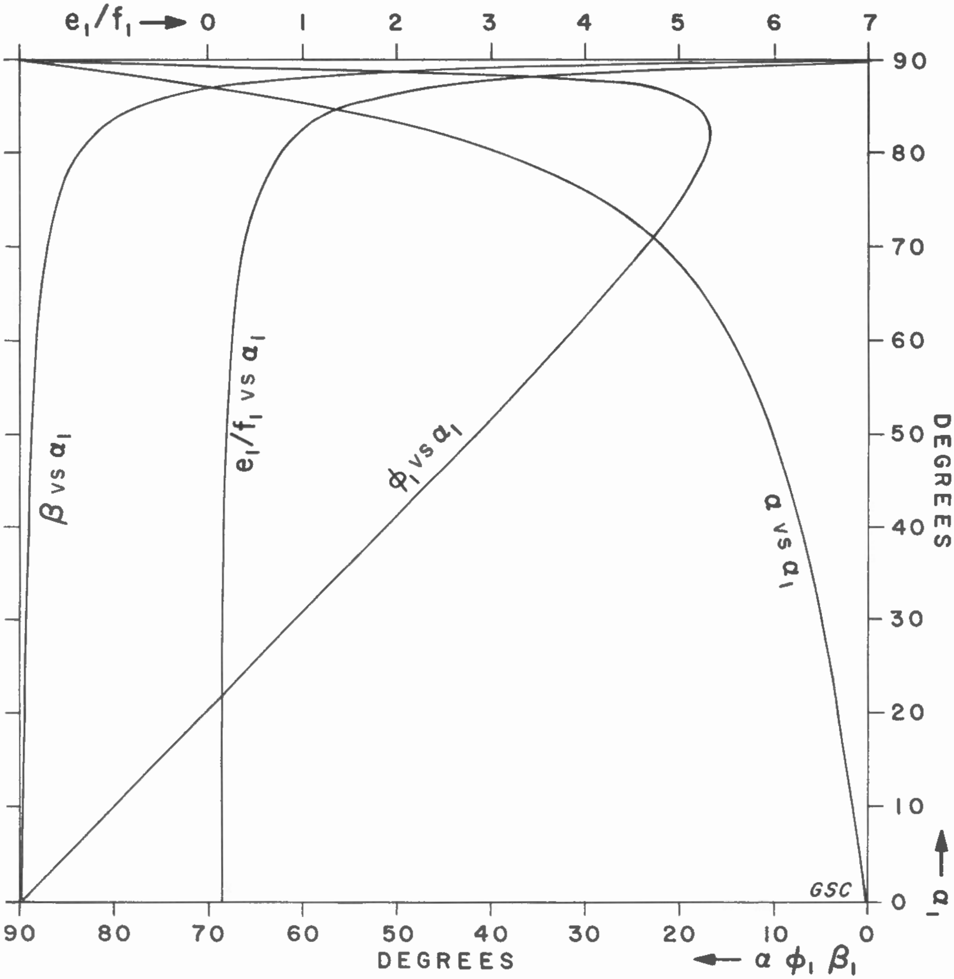

Figure 14. Nomograph for the determination of ϕ_1, β_1, α and e_1/f_1 from α_1 for $X=7$

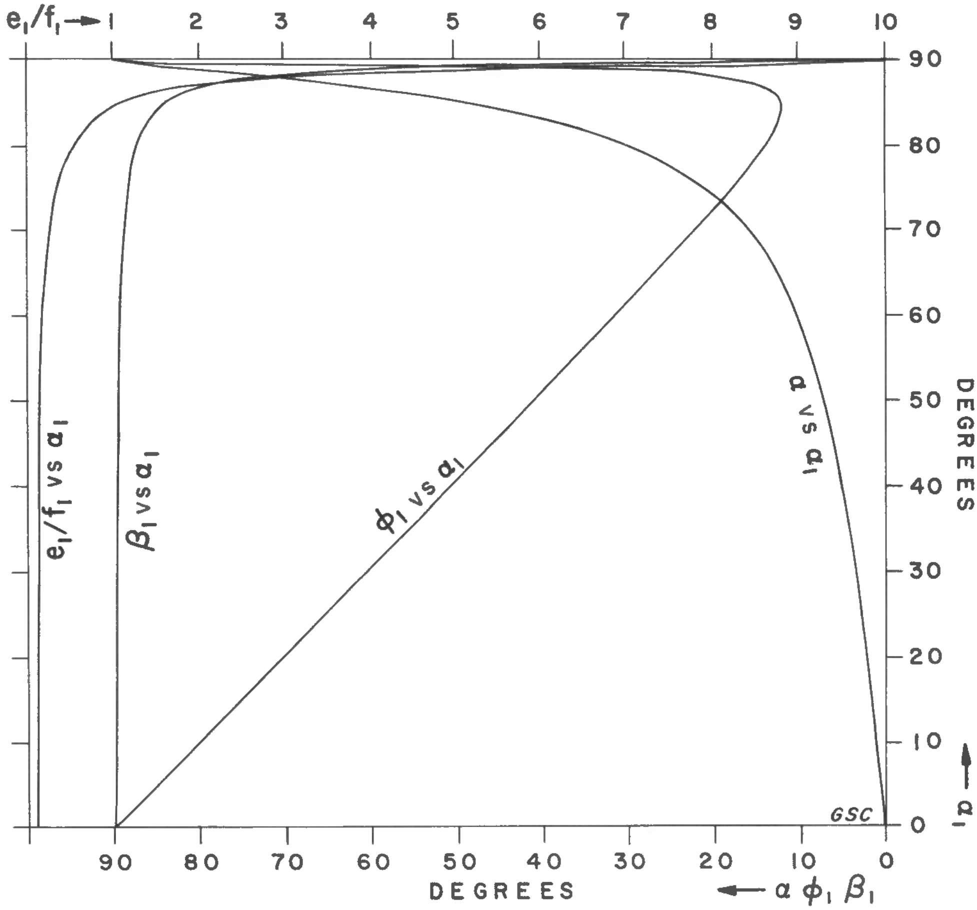

Figure 15. Nomograph for the determination of ϕ_1, β_1, α and e_1/f_1 from α_1 for $X = 10$

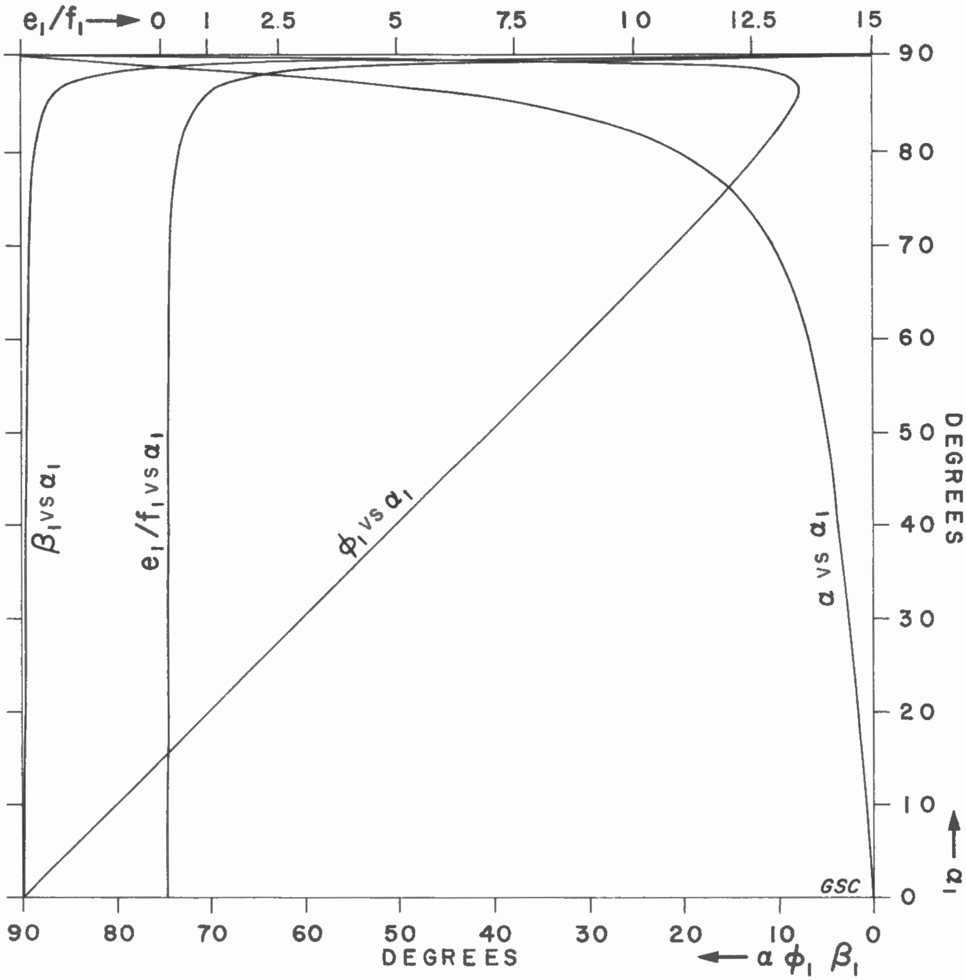

Figure 16. Nomograph for the determination of ϕ_1, β_1, α and e_1/f_1 from α_1 for X=15

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